Sums of sets of abelian group elements[☆]

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Abstract

For a positive integer k, let $f(k)$ denote the largest integer t such that for every finite abelian group G and every zero-sum free subset S of G, if $|S| = k$ then $|\Sigma(S)| \geq t$. In this paper, we prove that $f(k) \geq \frac{1}{6}k^2$, which significantly improves a result of J.E. Olson. We also supply some interesting results on $f(k)$.

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1. Introduction and Main Results

Let G be a finite abelian group and S be a sequence (or a subset) with elements of G. Let $\Sigma(S)$ denote the set of group elements which can be

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expressed as a sum of a nonempty subsequence (or a nonempty subset) of S. We say that S is zero-sum free if $0 \notin \Sigma(S)$.

For a positive integer k, let $f(G,k)$ denote the largest integer t such that $|\Sigma(S)| \geq t$ for every zero-sum free subset S of G with $|S| = k$. If G contains no such subset S, we set $f(G,k) = \infty$. Let

$$f(k) = \min_G f(G,k),$$

where G runs over all finite abelian groups.

The invariant $f(k)$ was first studied by R.B. Eggleton and P. Erdös in 1972 [3]. They determined the exact values of $f(k)$ for $k \leq 5$ and showed that $2k \leq f(k) \leq \lfloor \frac{k^2}{2} \rfloor + 1$ for $k \geq 4$. In 1975, J.E. Olson [14] proved that $f(k) \geq \frac{1}{9}k^2$, which is still the best known result on the lower bound of $f(k)$ for large $k(\geq 27)$. It was conjectured by R.B. Eggleton and P. Erdös [3] and proved by W. Gao et al. in 2008 [6] that $f(6) = 19$. In 2009, G. Bhowmik et al. [1] showed that $f(G,7) \geq 24$ for cyclic group G. Later P. Yuan and X. Zeng [24] extended the result to any finite abelian group and showed that $f(7) = 24$. Recently, J. Peng et al. [17] proved that $f(k) \geq 3k$ for $k \geq 6$. While the known upper bound $\lfloor \frac{k^2}{2} \rfloor + 1$ for $f(k)$ seems quite sharp, the lower bound $3k$ or $\frac{1}{9}k^2$ are far from ideal.

The main purpose of this paper is to improve the lower bound of $f(k)$. We state our main result as follows.

Theorem 1.1. *$f(k) \geq \frac{1}{6}k^2$ holds for every positive integer k.*

We will prove Theorem 1.1 by an inductive method, so we need to check the theorem for some small k first. To be more precise, we first verify the result for $1 \leq k \leq 28$, and then prove it for every k.

The associated inverse problem of $f(k)$ is to determine the structures of zero-sum free subsets S such that $|S| = k$ and $|\Sigma(S)| = f(k)$. The cases for $k = 1$ and $k = 2$ are trivial and the case when $k = 3$ is included in [9, Proposition 5.3.2]. In 2010, H. Guan et al. [11] described all the zero-sum free subsets S of an abelian group G such that $|S| = 5$ and $|\Sigma(S)| = 13$. Recently, J. Peng and W. Hui [16] gave the answers to the inverse problems of $f(k)$ when $k = 4$ and $k = 6$ (see Lemma 3.4).

Suppose S is a zero-sum free subset of a finite abelian group G with $|S| = 7$. Recently, J. Peng et al. [18] proved that if $\langle S \rangle$ is not cyclic, then $|\Sigma(S)| \geq 25$. This together with the result of G. Bhowmik et al. [1] allows J. Peng et al. [18] to obtain that if $|\Sigma(S)| = 24$ then $\langle S \rangle$ is a cyclic group and $25 \mid |\langle S \rangle|$. In this paper we improve this result to the following.

Theorem 1.2. *Let G be a finite abelian group and S be a zero-sum free subset of G such that $|S| = 7$. Then $|\Sigma(S)| = 24$ if and only if $\langle S \rangle$ is a cyclic group of order 25.*

Apart from being of interest in their own rights, the invariants $f(k)$ are useful tools in the investigation of various other problems in combinatorial and additive number theory.

Let $\mathsf{Ol}(G)$ denote the smallest positive integer t such that every subset S of G with length $|S| \geq t$ has a nonempty zero-sum subset. The invariant $\mathsf{Ol}(G)$ is called the *Olson constant* of G (see [15] for the most recent progress on the Olson constant). Clearly, the largest length of zero-sum free subset of G is $\mathsf{Ol}(G) - 1$. Therefore, if $f(G, k) \geq f(k) \geq \frac{1}{c}k^2$ for some $c \in \mathbb{R}_{>0}$ and every $k \in \mathbb{N}$, then $\mathsf{Ol}(G) < \sqrt{c|G|} + 1$ (see [9, Lemma 5.1.17] for details). So we have the following corollary of Theorem 1.1.

Corollary 1.3. $\mathsf{Ol}(G) < \sqrt{6|G|} + 1$ *for every finite abelian group G.*

On the other hand, the exact values of $\mathsf{Ol}(G)$ can be used to determine $f(G, k)$ and $f(k)$. In 1996, Y.O. Hamidoune and G. Zémor [12] proved that $\mathsf{Ol}(G) \leq \sqrt{2|G|} + \varepsilon(|G|)$ for some real value function of $\varepsilon(n) = O(n^{1/3} \ln n)$. It seems that the lower bound of $f(k)$ is tend to $\frac{k^2}{2}$. Based on some known values and our recent computation for $\mathsf{Ol}(G)$, we prove the following results.

Theorem 1.4. *The lower bounds of $f(k)$ for $1 \leq k \leq 28$ are stated in Table 1.*

k	$f(k) =$	k	$f(k) \geq$	k	$f(k) \geq$	k	$f(k) \geq$
1	1	8	30	15	69	22	96
2	3	9	35	16	71	23	102
3	5	10	41	17	73	24	108
4	8	11	47	18	74	25	115
5	13	12	54	19	80	26	122
6	19	13	61	20	85	27	127
7	24	14	66	21	91	28	132

Table 1: Lower bound of $f(k)$

As a corollary of Theorem 1.4, we have the following results.

Corollary 1.5. *1.* $f(k) \geq \lfloor \frac{1}{2}k^2 \rfloor$ *for* $k \leq 7$.

2. $f(k) \geq \frac{1}{3}k^2$ *for* $k \leq 14$.

3. $f(k) \geq \frac{1}{4}k^2$ *for* $k \leq 17$.

4. $f(k) \geq \frac{1}{5}k^2$ *for* $k \leq 21$.

5. $f(k) \geq \frac{1}{6}k^2$ *for* $k \leq 28$.

A further application of $f(k)$ deals with the study of the structure of long zero-sum free sequences. This is a topic going back to J.D. Bovey, P. Erdös and I. Niven [2] which found a lot of interest in recent years (see contributions by Gao, Geroldinger, Hamidoune, Savchev, Chen and others [4, 10, 19, 20, 21, 7, 23]). Based on the results of Theorem 1.4, we obtain the following.

Theorem 1.6. *Let* G *be a cyclic group of order* n*. If* S *is a zero-sum free sequence over* G *of length* $|S| \geq \frac{14n+152}{66}$*, then* S *contains some element with multiplicity at least* $\frac{7|S|-n+1}{32}$.

The paper is organized as follows. Section 2 provides some notations and concepts which will be used in the sequel. In section 3 we list some results on the inverse problem of $f(k)$ and provide a proof of Theorem 1.2. Section 4 deals with the lower bounds on $f(k)$ for $k \leq 28$. In Section 5 we prove Theorem 1.1. In the last Section we give a proof for Theorem 1.6.

2. Notations and Preliminaries

2.1. Notations

Our notation and terminology are consistent with [5, 8, 9]. Let $\mathbb{N}$ and $\mathbb{Z}$ be the sets of positive integers and all integers respectively, and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For $a, b \in \mathbb{Z}$ we set $[a, b] = \{x \in \mathbb{Z} | a \leq x \leq b\}$.

Let G be an additive finite abelian group and let C_n denote the cyclic group of order n. Let $\mathrm{ord}(g)$ denote the order of $g \in G$. Let $\mathcal{F}(G)$ denote the multiplicative, free abelian monoid with basis G. The elements of $\mathcal{F}(G)$ are called *sequences* over G. Every sequence S over G can be written in the form

$$S = g_1 \cdot \ldots \cdot g_\ell = \prod_{g \in G} g^{\mathsf{v}_g(S)},$$

where $\mathsf{v}_g(S) \in \mathbb{N}_0$ denotes the *multiplicity* of g in S. If $\mathsf{v}_g(S) \le 1$ for all $g \in G$, we call S a *subset* of G. We note that a subset S of G is always regarded as a special sequence over G.

We call $\mathrm{supp}(S) = \{g \in G \mid \mathsf{v}_g(S) > 0\}$ the *support* of S, $\mathsf{h}(S) = \max\{\mathsf{v}_g(S) \mid g \in G\}$ the *maximum of the multiplicity* in S, $|S| = \ell = \sum_{g\in G} \mathsf{v}_g(S) \in \mathbb{N}_0$ the *length* of S, and $\sigma(S) = \sum_{i=1}^{\ell} g_i = \sum_{g\in G} \mathsf{v}_g(S)g \in G$ the *sum* of S.

A sequence T is called a *subsequence* of S if $\mathsf{v}_g(T) \le \mathsf{v}_g(S)$ for all $g \in G$. Whenever T is a subsequence of S, let ST^{-1} denote the subsequence with T deleted from S. If S_1, S_2 are two sequences over G, let S_1S_2 denote the sequence over G satisfying that $\mathsf{v}_g(S_1S_2) = \mathsf{v}_g(S_1) + \mathsf{v}_g(S_2)$ for all $g \in G$. Let

$$\Sigma(S) = \{\sigma(T) \mid T \text{ is a subsequence of } S \text{ with } 1 \le |T| \le |S|\}.$$

The sequence S is called *zero-sum* if $\sigma(S) = 0 \in G$ and *zero-sum free* if $0 \notin \Sigma(S)$. If $\sigma(S) = 0$ and $\sigma(T) \ne 0$ for every subsequence T of S with $1 \le |T| < |S|$, then S is called *minimal zero-sum*.

For a subgroup H of G, let $\varphi : G \to G/H$ denote the canonical epimorphism. For a sequence $S = g_1 \cdot \ldots \cdot g_\ell$ over G, let $\varphi(S)$ denote the sequence $\varphi(g_1) \cdot \ldots \cdot \varphi(g_\ell)$ over G/H.

2.2. Some basic results

We first list the known values of $f(k)$, which can be found in [3, 6, 24].

Lemma 2.1.

(1) $f(k) \ge 2k - 1$, *and the equality holds if and only if* $k \in [1, 3]$.

(2) $f(4) = 8$.

(3) $f(5) = 13$.

(4) $f(6) = 19$.

(5) $f(7) = 24$.

We also need the following.

Lemma 2.2. *[9, Theorem 5.3.1] Let G be a finite abelian group and let $S = S_1 \cdot \ldots \cdot S_t$ be a zero-sum free sequence over G, where $S_1, \ldots, S_t$ are subsequences of S. Then*

$$|\Sigma(S)| \ge |\Sigma(S_1)| + \ldots + |\Sigma(S_t)|.$$

Lemma 2.3. *[6, Theorem 3.2] Let G be a finite abelian group and let S be a zero-sum free subset of G of length $|S| \in [4,7]$. If S contains an element of order 2, then*

$$|\Sigma(S)| \geq \lfloor \frac{1}{2}|S|^2 \rfloor + 1.$$

Lemma 2.4. *Let $S = x_1 \cdot x_2 \cdot \ldots \cdot x_k$ be a zero-sum free subset of a finite abelian group G such that $\mathrm{ord}(x_1) = \ldots = \mathrm{ord}(x_t) = 2$ for some $t \in [1,k]$ and let $H = \langle x_1, \ldots, x_t \rangle$. Let $\varphi : G \to G/H$ denote the canonical epimorphism and $T = \varphi(x_{t+1}) \cdot \ldots \cdot \varphi(x_k)$. Then*

(1) T is a zero-sum free sequence over G/H;

(2) $\mathsf{v}_g(T) \leq 2^t$ for every $g \in G/H$;

(3) $|\Sigma(S)| = 2^t - 1 + 2^t|\Sigma(T)|$;

(4) $|\Sigma(S)| \geq 2^t(k - t + 1) - 1$.

Proof. (1). We first show that T is zero-sum free. Suppose that there exists a nonempty subsequence T_1 of T such that $\sigma(T_1) = 0 \in G/H$. Then there exists a subset S_1 of $x_{t+1} \cdot \ldots \cdot x_k$ such that $T_1 = \varphi(S_1)$ and $\sigma(S_1) \in H$. Since S is zero-sum free, we have $\sigma(S_1) = h \in H \setminus \{0\}$. Note that $\Sigma(x_1 \cdot \ldots \cdot x_t) = H \setminus \{0\}$ and $\mathrm{ord}(h) = 2$. We can find a subset V of $x_1 \cdot \ldots \cdot x_t$ such that $\sigma(V) = h$, and then $V \cdot S_1$ is a zero-sum subset of S, yielding a contradiction. Therefore, T is zero-sum free and (1) holds.

(2). If $|T| = k - t \leq 2^t$, there is nothing to prove. Next assume that $k - t > 2^t$. Assume to the contrary that

$$\varphi(x_{j_1}) = \varphi(x_{j_2}) = \ldots = \varphi(x_{j_{2^t+1}}),$$

where $t + 1 \leq j_1 < j_2 < \ldots < j_{2^t+1} \leq k$. Then

$$\varphi(x_{j_2} - x_{j_1}) = \ldots = \varphi(x_{j_{2^t+1}} - x_{j_1}) = 0 \in G/H,$$

and therefore, $x_{j_2} - x_{j_1}, \ldots, x_{j_{2^t+1}} - x_{j_1} \in H$. Since S is a subset of G, we have that $x_{j_2} - x_{j_1}, \ldots, x_{j_{2^t+1}} - x_{j_1}$ are pairwise distinct. Therefore, there exists $m \in [2, 2^t + 1]$ such that $x_{j_m} - x_{j_1} = 0$, yielding a contradiction to that S is a subset. This proves (2).

(3). Let $\Sigma(T) = \{\overline{y_1}, \overline{y_2}, \ldots, \overline{y_r}\}$, where $r = |\Sigma(T)|$ and $\overline{y_i} = y_i + H \in G/H$ for every $i \in [1,r]$. Then $\Sigma(S) = \Sigma(x_1 \cdot \ldots \cdot x_t) \cup (y_1 + H) \cup \ldots \cup (y_r + H)$ is a disjoint union. Therefore, $|\Sigma(S)| = 2^t - 1 + 2^t|\Sigma(T)|$ and (3) holds.

(4). By Lemma 2.2, we have $|\Sigma(T)| \geq |T|$, and thus $|\Sigma(S)| \geq 2^t(k-t+1)-1$.

This completes the proof.

2.3. Olson's techniques

Let G be a finite abelian group, B be a subset of G, and x be an element of G. Following Olson [13], we write

$$\lambda_B(x) = |(B+x) \cap (G \setminus B)| = |(B+x) \setminus B|.$$

Lemma 2.5. *[13, 14] Let B and C be subsets of a finite abelian group G such that $0 \notin C$. Then for all $x, y \in G$, we have*

(1) $\lambda_B(x) = \lambda_{G\setminus B}(x)$.

(2) $\lambda_B(x) = \lambda_B(-x)$.

(3) $\lambda_B(x+y) \leq \lambda_B(x) + \lambda_B(y)$.

(4) $\sum_{x\in C} \lambda_B(x) \geq |B|(|C| - |B| + 1)$.

Lemma 2.6. *[13] Let G be a finite abelian group. Let S be a subset of G such that $0 \notin S$. Then for every $x \in S$ we have*

$$|\Sigma(S)| \geq |\Sigma(Sx^{-1})| + \lambda_B(x),$$

where $B = \Sigma(S)$.

The following result is exactly Lemma 3.1 of [14].

Lemma 2.7. *Let B and S be subsets of G such that $0 \notin S$ and let $H = \langle S \rangle$. Suppose $|H| \geq 2\min\{|B|, |G \setminus B|\}$. Then there is an $x \in S$ such that*

$$\lambda_B(x) \geq \min(\frac{|B|+1}{2}, \frac{|G \setminus B|+1}{2}, \frac{|S \cup (-S)|+2}{4}).$$

If A is a subset of a finite abelian group G and n is a positive integer, let $nA = A + \ldots + A$ (n times). The following result is also due to [14].

Lemma 2.8. *Let G be a finite abelian group, A be a subset of G with $0 \in A$, and n be a positive integer. Then either $nA = \langle A \rangle$ or $|nA| \geq |A| + (n-1)\lfloor \frac{1}{2}(|A|+1) \rfloor$.*

3. On the inverse problem of $f(k)$

In this section we list some results on the inverse problem of $f(k)$ and prove Theorem 1.2. Let P_n denote the symmetric group on $[1, n]$.

Lemma 3.1. *[9, Proposition 5.3.2] Let G be a finite abelian group and let $S = x_1 \cdot x_2 \cdot x_3$ be a zero-sum free subset of G. Then $|\Sigma(S)| = 5$ if and only if there exists $\tau \in P_3$ such that $\mathrm{ord}(x_{\tau(1)}) = 2$ and $x_{\tau(3)} = x_{\tau(1)} + x_{\tau(2)}$.*

Lemma 3.2. *[16] Let G be a finite abelian group and let T be a zero-sum free subset of G of length $|T| = 4$. Then $|\Sigma(T)| = 8$ if and only if there exists $x \in G$ such that $T = x \cdot (3x) \cdot (4x) \cdot (7x)$ and $\mathrm{ord}(x) = 9$.*

Lemma 3.3. *[11] Let G be a finite abelian group and let S be a zero-sum free subset of G of length $|S| = 5$. Then $|\Sigma(S)| = 13$ if and only if there exist $x_1, x_2 \in G$ such that S is one of the following forms:*

(i) $S = (-2x_1) \cdot x_1 \cdot (3x_1) \cdot (4x_1) \cdot (5x_1)$, *where* $\mathrm{ord}(x_1) = 14$.

(ii) $S = x_1 \cdot x_2 \cdot (x_1 + x_2) \cdot (2x_2) \cdot (x_1 + 2x_2)$, *where* $\mathrm{ord}(x_1) = 2$.

Lemma 3.4. *[16] Let G be a finite abelian group and let S be a zero-sum free subset of G of length $|S| = 6$. Then $|\Sigma(S)| = 19$ if and only if there exist $x_1, x_2, x_3 \in G$ such that S is one of the following forms:*

(i) $S = x_1 \cdot x_2 \cdot x_3 \cdot (x_1 + x_3) \cdot (x_2 + x_3) \cdot (x_1 + x_2 + x_3)$, *where* $\mathrm{ord}(x_1) = 2$ *and* $2x_2 \in \langle x_1 \rangle$.

(ii) $S = x_1 \cdot x_2 \cdot (2x_2) \cdot (3x_2) \cdot (x_1 + x_2) \cdot (x_1 + 2x_2)$, *where* $\mathrm{ord}(x_1) = 2$.

(iii) $S = (-2x_1) \cdot x_1 \cdot (3x_1) \cdot (4x_1) \cdot (5x_1) \cdot (6x_1)$, *where* $\mathrm{ord}(x_1) = 20$.

(iv) $S = (-3x_1) \cdot x_1 \cdot (4x_1) \cdot (5x_1) \cdot (9x_1) \cdot (12x_1)$, *where* $\mathrm{ord}(x_1) = 20$.

(v) $S = x_1 \cdot x_2 \cdot (x_1 + x_2) \cdot (x_1 + 2x_2) \cdot (2x_1 + x_2) \cdot (4x_1 + 4x_2)$, *where* $2x_1 = 2x_2, \mathrm{ord}(x_1) = \mathrm{ord}(x_2) = 10$.

Lemma 3.5. *[18, Theorem 1.2] Let G be an abelian group and S be a zero-sum free subset of G of length $|S| = 7$. If $|\Sigma(S)| = 24$, then $\langle S \rangle$ is a cyclic group and $25 \mid |\langle S \rangle|$.*

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. Since $f(7) = 24$, we have that $|\Sigma(S)| \geq 24$. It remains to show that if $|\Sigma(S)| = 24$ then $\langle S\rangle \cong C_{25}$.

Assume to the contrary that $\langle S\rangle \ncong C_{25}$. By Lemma 3.5 we obtain that $\langle S\rangle$ is a cyclic group and $|\langle S\rangle| \geq 50$. It follows from Lemma 2.3 that S contains no elements of order 2.

We assert that for every $x \in S$, $|\Sigma(Sx^{-1})| \geq 20$. Otherwise, there exists an $x_0 \in S$ such that $|\Sigma(Sx_0^{-1})| \leq 19$. Since $|Sx_0^{-1}| = 6$ and $f(6) = 19$, we have that $|\Sigma(Sx_0^{-1})| = 19$. Since S contains no elements of order 2, we obtain that there exists $x_1, x_2 \in G$ such that Sx_0^{-1} is of form (iii) or (iv) or (v) in Lemma 3.4. If Sx_0^{-1} is of form (v), we infer that $\mathrm{ord}(x_1 - x_2) = 2$ and $\langle Sx_0^{-1}\rangle = \langle x_1 - x_2, x_2\rangle \cong C_2 \oplus C_{10}$, yielding a contradiction to that $\langle S\rangle$ is a cyclic group. Therefore, Sx_0^{-1} is of form (iii) or (iv) in Lemma 3.4. In these cases we infer that $\Sigma(Sx_0^{-1}) = \langle Sx_0^{-1}\rangle \setminus \{0\}$. Since S is zero-sum free, we have $x_0 \notin \langle Sx_0^{-1}\rangle$ and thus $\Sigma(S) = \Sigma(Sx_0^{-1}) \cup \{x_0\} \cup (\Sigma(Sx_0^{-1}) + x_0)$ is a disjoint union. Therefore, $|\Sigma(S)| = 2|\Sigma(Sx_0^{-1})| + 1 = 39$, yielding a contradiction. This proves our assertion. Hence for every $x \in S$, $|\Sigma(Sx^{-1})| \geq 20$.

Let $B = \Sigma(S)$. By Lemma 2.6, we have $|\Sigma(S)| \geq |\Sigma(Sx^{-1})| + \lambda_B(x)$ for every $x \in S$. Therefore, $\lambda_B(x) \leq |\Sigma(S)| - |\Sigma(Sx^{-1})| \leq 4$ for every $x \in S$.

Since S contains no elements of order 2, we infer that $|S \cup (-S)| = 14$. Choose $y \in S$ such that $\lambda_B(y) = \max\{\lambda_B(x), x \in S\}$. Applying Lemma 2.7 to $H = \langle S\rangle$, we obtain that

$$\begin{aligned}
\lambda_B(y) &\geq \min\Big(\frac{|B|+1}{2}, \frac{|G \setminus B|+1}{2}, \frac{|S \cup (-S)|+2}{4}\Big) \\
&\geq \min\Big(\frac{|B|+1}{2}, \frac{|\langle S\rangle \setminus B|+1}{2}, \frac{|S \cup (-S)|+2}{4}\Big) \\
&\geq \min\Big(\frac{24+1}{2}, \frac{26+1}{2}, \frac{14+2}{4}\Big) = 4.
\end{aligned}$$

Therefore, $\lambda_B(y) = 4$.

Now let $k = \min(|B|, |\langle S\rangle \setminus B|)$, $A = S \cup (-S) \cup \{0\}$, and $u = \lfloor \frac{1}{2}(|A|+1)\rfloor = 8$. Then $k = |B| = 24$, $\langle A\rangle = \langle S\rangle$, and $|\langle A\rangle| \geq 50 > 48 = 2k$. By Lemma 2.8, we have that either

$$|nA| \geq |\langle A\rangle| \geq 50 > 48 = 2k$$

or

$$|nA| \geq |A| + (n-1)\lfloor \frac{1}{2}(|A|+1)\rfloor = 15 + u(n-1)$$

for every integer n. Let $2k = 48 = 15 + (r-2)u + q$, where $0 \le q < u = 8$. Therefore, $r = 6$ and $q = 1$ and

$$\begin{aligned} |6A| &\ge \min(|\langle A\rangle|, 15 + (r-1)u) \\ &= \min(|\langle A\rangle|, 15 + (6-1)\times 8) > 48. \end{aligned}$$

Since $0 \in A$, we infer that $A \subset 2A \subset \ldots \subset 6A$. Now we can choose a subset C of $6A \setminus \{0\}$ such that $|C| = 47$ and $|nA \cap C| \ge 14 + (n-1)u$ for each $1 \le n \le 5$. So there exist pairwise disjoint subsets $A_1, \ldots, A_6$ such that $6A \cap C = A_1 \cup \ldots \cup A_6$, $|A_1| = 14$, $|A_2| = \ldots = |A_5| = 8$, $|A_6| = 1$, and $A_n \subset nA \cap C$ for all $n \in [1, 6]$. It follows from Lemma 2.5 that if $c \in A_n$, then $\lambda_B(c) \le n\lambda_B(y) = 4n$ for every $n \in [1, 6]$. Therefore,

$$\begin{aligned} \sum_{c\in C} \lambda_B(c) &= \sum_{c\in A_1} \lambda_B(c) + \sum_{c\in A_2} \lambda_B(c) + \ldots + \sum_{c\in A_6} \lambda_B(c) \\ &\le 4|A_1| + 8|A_2| + \ldots + 24|A_6| = 528. \end{aligned}$$

On the other hand, by Lemma 2.5, we infer that

$$\sum_{c\in C} \lambda_B(c) \ge |B|(|C| - |B| + 1) = 24 \times (47 - 24 + 1) = 576,$$

which yields a contradiction. Therefore, $\langle S\rangle \cong C_{25}$, and we are done.

This completes the proof.

By using a computer program, we obtain that if S is a zero-sum free subset of C_{25} of length $|S| = 7$, then there exists $g \in C_{25}$ such that $S = g \cdot (5g) \cdot (6g) \cdot (10g) \cdot (11g) \cdot (16g) \cdot (21g)$ and $\mathrm{ord}(g) = 25$. This together with Theorem 1.2 implies the following result.

Corollary 3.6. *Let G be a finite abelian group and S be a zero-sum free subset of G with $|S| = 7$. Then the following statements are equivalent.*

(1) $|\Sigma(S)| = 24$.

(2) $\langle S\rangle$ *is a cyclic group of order* 25.

(3) $\Sigma(S) = \langle S\rangle \setminus \{0\}$ *where* $\langle S\rangle \cong C_{25}$.

(4) There exists $g \in G$ *such that* $S = g \cdot (5g) \cdot (6g) \cdot (10g) \cdot (11g) \cdot (16g) \cdot (21g)$ *and* $\mathrm{ord}(g) = 25$.

4. On the lower bounds of $f(k)$ for small k

In 2000, J. Subocz [22] supplied a table with the values of $\mathsf{Ol}(G)$ for all abelian groups G with order $|G| \le 55$ and all cyclic groups G with order $|G| \le 64$. By using some computer programs, we are able to extend the table of J. Subocz to the following.

G	$\mathsf{Ol}(G)$	G	$\mathsf{Ol}(G)$	G	$\mathsf{Ol}(G)$
$\|G\| \le 33$	≤ 8	C_{63}	11	C_{66}	12
$\|G\| \le 41$	≤ 9	$C_3 \oplus C_{21}$	11	C_{67}	12
$\|G\| \le 51$	≤ 10	C_{64}	12	C_{68}	12
$\|G\| \le 55$	≤ 11	$C_2 \oplus C_{32}$	12	$C_2 \oplus C_{34}$	12
C_{56}	11	$C_4 \oplus C_{16}$	12	C_{69}	12
$C_2 \oplus C_{28}$	11	$C_2^2 \oplus C_{16}$	12	C_{70}	12
$C_2^2 \oplus C_{14}$	11	C_8^2	11	C_{71}	12
C_{57}	11	$C_2 \oplus C_4 \oplus C_8$	11	C_{72}	12
C_{58}	11	$C_2^3 \oplus C_8$	11	$C_2 \oplus C_{36}$	12
C_{59}	11	C_4^3	9	$C_3 \oplus C_{24}$	12
C_{60}	11	$C_2^2 \oplus C_4^2$	9	$C_6 \oplus C_{12}$	12
$C_2 \oplus C_{30}$	11	$C_2^4 \oplus C_4$	8	$C_2 \oplus C_6^2$	11
C_{61}	11	C_2^6	7	$C_2^2 \oplus C_{18}$	12
C_{62}	12	C_{65}	12	C_{73}	12

Table 2: $\mathsf{Ol}(G)$ for abelian groups.

By Table 2, we obtain the following.

Lemma 4.1. *Let G be a finite abelian group and let S be a zero-sum free subset of G. Then*

(1) If $|S| = 8$, then $|\langle S\rangle| \ge 34$.

(2) If $|S| = 9$, then $|\langle S\rangle| \ge 42$.

(3) If $|S| = 10$, then $|\langle S\rangle| \ge 52$.

(4) If $|S| = 11$, then $|\langle S\rangle| \ge 62$.

(5) If $|S| \ge 12$, then $|\langle S\rangle| \ge 74$.

Lemma 4.2. *Let G be a finite abelian group and let S be a zero-sum free subset of G with an element of order 2. Then*

(1) If $|S| = 8$, then $|\Sigma(S)| \geq 31$.

(2) If $|S| = 9$, then $|\Sigma(S)| \geq 37$.

(3) If $|S| = 10$, then $|\Sigma(S)| \geq 43$.

(4) If $|S| = 11$, then $|\Sigma(S)| \geq 53$.

(5) If $|S| = 12$, then $|\Sigma(S)| \geq 65$.

(6) If $|S| \geq 13$, then $|\Sigma(S)| \geq 77$.

Proof. Let $S = x_1 \cdot x_2 \cdot \ldots \cdot x_k$, with $k \geq 8$ and $\mathrm{ord}(x_1) = 2$, and let $H = \langle x_1 \rangle = \{0, x_1\}$. Let $\varphi : G \to G/H$ denote the canonical epimorphism and $T = \varphi(x_2) \cdot \ldots \cdot \varphi(x_k)$. It follows from Lemma 2.4 that T is a zero-sum free sequence and $\mathsf{v}_g(T) \leq 2$ for every $g \in G/H$. Therefore, $|\mathrm{supp}(T)| \geq 4$.

We now prove the lemma for the case when $|S| = 8$. The proofs of other cases are similar and we omit them here. By Lemma 2.4, we have $|\Sigma(S)| = 1 + 2|\Sigma(T)|$. It suffices to show that $|\Sigma(T)| \geq 15$.

If $|\mathrm{supp}(T)| \geq 5$, then we can write T as $T = T_1 \cdot T_2$, where T_1 and T_2 are subsets of G/H with $|T_1| = 5$ and $|T_2| = 2$. It follows from Lemmas 2.2 and 2.1 that $|\Sigma(T)| \geq |\Sigma(T_1)| + |\Sigma(T_2)| \geq f(5) + f(2) = 16 \geq 15$, and we are done.

Next we assume that $|\mathrm{supp}(T)| = 4$ and T is of form $a^2b^2c^2d$. Let $T_1 = abcd$ and $T_2 = abc$. Since T is zero-sum free, we obtain that T_2 contains no elements of order 2. By Lemma 2.1 and Lemma 3.1, we have $|\Sigma(T_2)| \geq 6$. Note that $|\Sigma(T_1)| \geq f(4) = 8$. If $|\Sigma(T_1)| \geq 9$, then by Lemma 2.2, $|\Sigma(T)| \geq |\Sigma(T_1)| + |\Sigma(T_2)| \geq 9 + 6 = 15$, and we are done. So we may assume that $|\Sigma(T_1)| = 8$. Now by Lemma 3.2, there exists $y \in G/H$ such that $T_1 = y \cdot 3y \cdot 4y \cdot 7y$ and $\mathrm{ord}(y) = 9$. Therefore, $T = (i_1y) \cdot (i_2y) \cdot (i_3y) \cdot y \cdot (3y) \cdot (4y) \cdot (7y)$, where $\{i_1, i_2, i_3\} \subset \{1, 3, 4, 7\}$. This is impossible since T is zero-sum free.

This completes the proof.

We also need the following lemma which can be easily checked by computer programs.

Lemma 4.3. *$f(C_{34}, 8) = 33$, $f(C_{35}, 8) = 34$, $f(C_{36}, 8) = 33$, $f(C_2 \oplus C_{18}, 8) = 33$, $f(C_3 \oplus C_{12}, 8) = 35$, and $f(C_6 \oplus C_6, 8) = 35$.*

We are now in the position to prove Theorem 1.4.

PROOF OF THEOREM 1.4. Let G be a finite abelian group and S be a zero-sum free subset of G such that $|\Sigma(S)| = f(k)$. Without loss of generality we may assume that $G = \langle S \rangle$.

Suppose $k = 8$. Assume to the contrary that $|\Sigma(S)| = f(8) \le 29$. By Lemmas 4.1, 4.3, and 4.2, we obtain that $|G| \ge 37$ and S contains no elements of order 2. Clearly $|\Sigma(S)| = f(8) > f(7) = 24$.

We assert that for every $x \in S$, $|\Sigma(Sx^{-1})| \ge 25$. Otherwise, there exists an $x_0 \in S$ such that $|\Sigma(Sx_0^{-1})| \le 24$. Since $f(7) = 24$, we have that $|\Sigma(Sx_0^{-1})| = 24$. Then by Corollary 3.6, we infer that $\Sigma(Sx_0^{-1}) = \langle Sx_0^{-1} \rangle \setminus \{0\}$. Since S is zero-sum free, we have $x_0 \notin \langle Sx_0^{-1} \rangle$ and thus $\Sigma(S) = \Sigma(Sx_0^{-1}) \cup \{x_0\} \cup (\Sigma(Sx_0^{-1}) + x_0)$ is a disjoint union. Therefore, $|\Sigma(S)| = 2|\Sigma(Sx_0^{-1})| + 1 = 49$, yielding a contradiction. This proves our assertion. Hence for every $x \in S$, $|\Sigma(Sx^{-1})| \ge 25$.

Now let $B = \Sigma(S)$. Then $|G \setminus B| \ge |G| - |B| \ge 37 - 29 = 8$. Note that $|S \cup (-S)| \ge 16$. Applying Lemma 2.7 to $H = G$, we obtain that there exists $x \in S$ such that

$$\lambda_B(x) \ge \min\left(\frac{|B|+1}{2}, \frac{|G \setminus B|+1}{2}, \frac{|S \cup (-S)|+2}{4}\right) \ge \frac{9}{2}.$$

Therefore, $\lambda_B(x) \ge 5$.

Now by Lemma 2.6, we obtain that

$$|\Sigma(S)| \ge |\Sigma(Sx^{-1})| + \lambda_B(x) \ge 25 + 5 = 30,$$

yielding a contradiction. Therefore, $f(8) \ge 30$.

Similarly, we can show the lower bounds of $f(k)$ for $k \in [9, 17]$.

It follows from Lemma 2.2 that $f(m+n) \ge f(m) + f(n)$ for all positive integers $m, n \in \mathbb{N}$. Therefore, $f(18) \ge f(13) + f(5) \ge 74$. Similarly, we can get the lower bounds of $f(k)$ for $k \in [19, 28]$.

This completes the proof.

5. Proof of Theorem 1.1

In this section we will prove Theorem 1.1. We need some technical results. Let $\Sigma_0(S) = \{0\} \cup \Sigma(S)$.

Lemma 5.1. *Let G be a finite abelian group and let S be a zero-sum free generating subset of G such that $S \cap (-S) = \emptyset$ and $|S| = s \geq 25$. Then there exist a subset $T \subset S$ and integers $u, v, q \in [1, s]$ satisfying $3 \leq u \leq q \leq s$, $1 \leq v \leq q$, $|T| = s - v$, and*

$$5\left(\frac{3}{2}\right)^{u-3} + u - 2 < s \leq 5\left(\frac{3}{2}\right)^{u-2} + u - 1, \tag{1}$$

such that

$$|\Sigma(S)| \geq \begin{cases} \frac{1}{4}(s+1)(s-2) - \Delta, & \text{if } q = s; \\ \Omega + 1 + 2|\Sigma(T)|, & \text{if } q < s. \end{cases}$$

where

$$\Delta = \frac{1}{4}(2s - u + 1)(u - 2) - 5\left(\frac{3}{2}\right)^{u-2} + 2,$$

and

$$\Omega = \begin{cases} 0, & \text{if } v = 1; \\ 1, & \text{if } v = 2; \\ 5\left(\frac{3}{2}\right)^{v-3} - 2, & \text{if } 2 \leq v - 1 \leq u; \\ \frac{1}{4}(2s - v + 2)(v - 3) - \Delta, & \text{if } v - 1 > u. \end{cases}$$

Proof. Let $S = a_1 \cdot a_2 \cdot \ldots \cdot a_s$. We first arrange the elements of S as follows. Choose $a_1 \in S$ arbitrarily. After choosing $a_1, \ldots, a_{j-1}$, choose a_j from $S(a_1 \cdot \ldots \cdot a_{j-1})^{-1}$ such that the size of $\Sigma_0(a_1 \cdot \ldots \cdot a_j)$ is maximal (i.e. $|\Sigma_0(a_1 \cdot \ldots \cdot a_j)| = \max_{j \leq k \leq s}\{|\Sigma_0(a_1 \cdot \ldots \cdot a_{j-1} \cdot a_k)|\}$).

For each $1 \leq t \leq s$, let $B_t = \Sigma_0(a_1 \cdot \ldots \cdot a_t)$ and $\sigma_t = |B_t| = |\Sigma(a_1 \cdot \ldots \cdot a_t)| + 1$. Since S is zero-sum free, by Lemma 2.1 (1) we infer that $\sigma_1 = 2$ and $\sigma_2 = 4$. Define $\sigma_0 = 1$.

For each $2 \leq t \leq s$, let

$$k_t = \min\{|B_{t-1}|, |G \setminus B_{t-1}|\} \text{ and } H_t = \langle a_t \cdot \ldots \cdot a_s \rangle.$$

Let q be the smallest index ($q \geq 2$) such that $|H_{q+1}| < 2k_{q+1}$, and take $q = s$ if the inequality never occurs.

Let $\ell \in [t, s]$. Since $\Sigma_0(a_1, \ldots, a_{t-1}, a_\ell) \supseteq B_{t-1} \cup ((B_{t-1} + a_\ell) \cap (G \setminus B_{t-1}))$, we infer that

$$|\Sigma_0(a_1, \ldots, a_{t-1}, a_\ell)| \geq \sigma_{t-1} + \lambda_{B_{t-1}}(a_\ell)$$

for each $\ell \in [t, s]$.

Note that $S \cap (-S) = \emptyset$. So $|\{a_t, \ldots, a_s\} \cup \{-a_t, \ldots, -a_s\}| = 2(s-t+1)$. By Lemma 2.7, there exists $b \in \{a_t, \ldots, a_s\}$ such that

$$\lambda_{B_{t-1}}(b) \geq \min\{\frac{1}{2}(k_t + 1), \frac{1}{2}(s - t + 2)\}$$

for every $t \leq q$. According to the way that a_i was arranged, we have that

$$\sigma_t \geq \sigma_{t-1} + \min\{\frac{1}{2}(k_t + 1), \frac{1}{2}(s - t + 2)\}. \tag{2}$$

We next show that (2) holds if k_t is replaced by $|B_{t-1}| = \sigma_{t-1}$. Suppose this is not true. Then $k_t = |G \setminus B_{t-1}| < |B_{t-1}|$ and also $s - t + 2 > k_t + 1$. Hence $s - t + 1 \geq k_t + 1 = |G \setminus B_{t-1}| + 1$. Since S is zero-sum free, it follows from Lemmas 2.2 and 2.1 that

$$\begin{aligned} |\Sigma(S)| &\geq |\Sigma(a_1 \cdot \ldots \cdot a_{t-1})| + |\Sigma(a_t \cdot \ldots \cdot a_s)| \\ &\geq |B_{t-1}| - 1 + 2(s - t + 1) - 1 \\ &\geq |B_{t-1}| + 2|G \setminus B_{t-1}| \\ &\geq |G|, \end{aligned}$$

yielding a contradiction. Thus

$$\sigma_t \geq \sigma_{t-1} + \min\{\frac{1}{2}(\sigma_{t-1} + 1), \frac{1}{2}(s - t + 2)\}$$

for every $t \leq q$. Now we define numbers $y_0, y_1, \ldots, y_s$ by the recursions. $y_0 = 1$, $y_1 = 2$, $y_2 = 4$, and (for $2 < t \leq q$)

$$y_t = y_{t-1} + \min\{\frac{1}{2}(y_{t-1} + 1), \frac{1}{2}(s - t + 2)\}. \tag{3}$$

Clearly $\sigma_t \geq y_t$ for every $0 \leq t \leq q$.

Let $u = u(s)$ be the largest integer in the interval $3 \leq u < q$ such that

$$\frac{1}{2}(y_{u-1} + 1) < \frac{1}{2}(s - u + 2).$$

Clearly $u < s$. Hence

$$\frac{1}{2}(y_u + 1) \geq \frac{1}{2}(s - (u + 1) + 2),$$

and therefore,

$$y_{u-1} + u - 1 < s \leq y_u + u. \tag{4}$$

Thus Equation (3) becomes

$$y_t = \begin{cases} \frac{1}{2}(3y_{t-1} + 1), & \text{if } 3 \leq t \leq u; \\ y_{t-1} + \frac{1}{2}(s - t + 2), & \text{if } u < t \leq q. \end{cases}$$

Hence

$$y_t = 5\left(\frac{3}{2}\right)^{t-2} - 1 \quad (\text{for } 2 \leq t \leq u), \tag{5}$$

and for $u < t \leq q$,

$$\begin{aligned} y_t &= y_u + \sum_{j=3}^{t} \frac{1}{2}(s - j + 2) - \sum_{j=3}^{u} \frac{1}{2}(s - j + 2) \\ &= y_u + \frac{1}{4}(2s - t + 1)(t - 2) - \frac{1}{4}(2s - u + 1)(u - 2) \\ &= \frac{1}{4}(2s - t + 1)(t - 2) - \Delta + 1, \end{aligned}$$

where

$$\begin{aligned} \Delta &= \frac{1}{4}(2s - u + 1)(u - 2) - y_u + 1 \\ &= \frac{1}{4}(2s - u + 1)(u - 2) - 5\left(\frac{3}{2}\right)^{u-2} + 2. \end{aligned}$$

It follows from (4) and (5) that

$$5\left(\frac{3}{2}\right)^{u-3} + u - 2 < s \leq 5\left(\frac{3}{2}\right)^{u-2} + u - 1,$$

and this proves (1).

If $q = s$, we can take $t = q = s$, then

$$|\Sigma(S)| = \sigma_s - 1 \geq y_s - 1 = \frac{1}{4}(s + 1)(s - 2) - \Delta,$$

and we are done.

If $q < s$, then $|H_{q+1}| < 2k_{q+1} \leq |G|$ and thus $H_{q+1} = \langle a_{q+1} \cdot \ldots \cdot a_s \rangle$ is a proper subgroup of G. Since $G = \langle S \rangle$, we infer that there exists $j \in [1, q]$

such that $a_j \notin H_{q+1}$. Let $1 \le v \le q$ be the largest index such that $a_v \notin H_{q+1}$. Then $H_{q+1} = \langle a_{v+1} \cdot \ldots \cdot a_s \rangle$ and thus

$$\begin{aligned} & \Sigma(a_v \cdot a_{v+1} \cdot \ldots \cdot a_s) \\ = \; & \Sigma(a_{v+1} \cdot \ldots \cdot a_s) \cup \{a_v\} \cup (a_v + \Sigma(a_{v+1} \cdot \ldots \cdot a_s)) \end{aligned}$$

is a disjoint union. Therefore,

$$|\Sigma(a_v \cdot a_{v+1} \cdot \ldots \cdot a_s)| = 1 + 2|\Sigma(a_{v+1} \cdot \ldots \cdot a_s)|.$$

Let $\Omega = y_{v-1} - 1$. It follows from Lemma 2.2 that

$$\begin{aligned} |\Sigma(S)| & \ge |\Sigma(a_1 \cdot \ldots \cdot a_{v-1})| + |\Sigma(a_v \cdot a_{v+1} \cdot \ldots \cdot a_s)| \\ & = \sigma_{v-1} - 1 + 1 + 2|\Sigma(a_{v+1} \cdot \ldots \cdot a_s)| \\ & \ge y_{v-1} - 1 + 1 + 2|\Sigma(a_{v+1} \cdot \ldots \cdot a_s)| \\ & = \Omega + 1 + 2|\Sigma(a_{v+1} \cdot \ldots \cdot a_s)|. \end{aligned}$$

Take $T = a_{v+1} \cdot \ldots \cdot a_s$, and we are done.

This completes the proof.

Lemma 5.2. *Let k, s, u be positive integers such that $3 \le u \le s \le k$, $k \ge 29$, $s > \frac{8}{9}k$, and $5\left(\frac{3}{2}\right)^{u-3} + u - 2 < s \le 5\left(\frac{3}{2}\right)^{u-2} + u - 1$. Then*

$$2^{k-s} - 1 + \frac{1}{4}(s+1)(s-2) - \Delta \ge \frac{1}{6}k^2 + \frac{225}{48},$$

where $\Delta = \frac{1}{4}(2s-u+1)(u-2) - 5\left(\frac{3}{2}\right)^{u-2} + 2$.

Proof. Let $N = 2^{k-s} - 1 + \frac{1}{4}(s+1)(s-2) - \Delta$.

Since $5\left(\frac{3}{2}\right)^{u-2} + u - 1 \ge s > \frac{8}{9}k > 25$, we infer that $u \ge 6$. We distinguish several cases according to the value of u.

Case 1. $u \ge 11$. Then

$$\begin{aligned} s & > u - 2 + \frac{10}{3}\left(\frac{3}{2}\right)^{u-2} = u - 2 + \frac{10}{3}\left(1 + \frac{1}{2}\right)^{u-2} \\ & = (u-2) + \frac{10}{3}\sum_{i=0}^{u-2}\binom{u-2}{i}\frac{1}{2^i} > (u-2) + \frac{10}{3}\sum_{i=1}^{9}\binom{u-2}{i}\frac{1}{2^i} \end{aligned}$$

$$\begin{aligned}
&= (u-2) + \frac{10}{3}(u-2)\sum_{i=1}^{9}\binom{u-3}{i-1}\frac{1}{i2^i} \\
&\geq (u-2) + \frac{10}{3}(u-2)\sum_{i=1}^{9}\binom{8}{i-1}\frac{1}{i2^i} \\
&> \frac{29}{2}(u-2) > 130,
\end{aligned}$$

and thus $u < \frac{2}{29}s+2$. Since $5\left(\frac{3}{2}\right)^{u-2}+u-1 \geq s$, we have that $5\left(\frac{3}{2}\right)^{u-2}-2 \geq s-u-1$. Therefore,

$$\begin{aligned}
\Delta &\leq \frac{1}{4}(2s-u+1)(u-2)-s+u+1 \\
&\leq \frac{1}{4}(2s-\frac{2}{29}s-1)(\frac{2}{29}s)-s+\frac{2}{29}s+3 \\
&= \frac{28}{29^2}s^2-\frac{55}{58}s+3.
\end{aligned}$$

Note that $2^{k-s}-1 \geq k-s$, $s > \frac{8}{9}k$, and $k \geq s > 130$. So

$$\begin{aligned}
N &\geq k-s+\frac{1}{4}(s+1)(s-2)-\Delta \\
&\geq k-s+\frac{1}{4}s^2-\frac{1}{4}s-\frac{1}{2}-\frac{28}{29^2}s^2+\frac{55}{58}s-3 \\
&= (\frac{1}{4}-\frac{28}{29^2})s^2-(\frac{5}{4}-\frac{55}{58})s+k-\frac{7}{2} \\
&> (\frac{1}{4}-\frac{28}{29^2})(\frac{8}{9}k)^2-(\frac{5}{4}-\frac{55}{58})(\frac{8}{9}k)+k-\frac{7}{2} \\
&\geq \frac{1}{6}k^2+\frac{225}{48},
\end{aligned}$$

and we are done.

Case 2. $u=10$. Then $k \geq s > 5\left(\frac{3}{2}\right)^{u-3}+u-2 > 93$ and $\Delta = \frac{1}{4}(2s-u+1)(u-2)-5\left(\frac{3}{2}\right)^{u-2}+2 = 4s-144.1445 < 4s-144$. Note that $2^{k-s}-1 \geq k-s$ and $s > \frac{8}{9}k$. So

$$\begin{aligned}
N &\geq k-s+(\frac{1}{4}s^2-\frac{1}{4}s-\frac{1}{2})-4s+144 \\
&= \frac{1}{4}s^2-\frac{21}{4}s+k+144-\frac{1}{2} > \frac{1}{4}(\frac{8}{9}k)^2-\frac{21}{4}(\frac{8}{9}k)+k+144-\frac{1}{2} \\
&\geq \frac{1}{6}k^2+\frac{225}{48},
\end{aligned}$$

and we are done.

Case 3. $u = 9$. Then $k \geq s > 5\left(\frac{3}{2}\right)^{u-3} + u - 2 > 63$ and $\Delta = \frac{1}{4}(2s-u+1)(u-2) - 5\left(\frac{3}{2}\right)^{u-2} + 2 < \frac{7}{2}s - 97$. Using the same argument as in Case 2, we have that $N \geq \frac{1}{6}k^2 + \frac{225}{48}$.

Case 4. $u = 8$. Then $k \geq s > 5\left(\frac{3}{2}\right)^{u-3} + u - 2 > 43$ and $\Delta = \frac{1}{4}(2s-u+1)(u-2) - 5\left(\frac{3}{2}\right)^{u-2} + 2 < 3s - 65$.

If $k - s \geq 4$, then $2^{k-s} \geq 4(k-s)$. Note that $s > \frac{8}{9}k$. So

$$\begin{aligned} N &\geq 4(k-s) - 1 + (\frac{1}{4}s^2 - \frac{1}{4}s - \frac{1}{2}) - 3s + 65 \\ &= \frac{1}{4}s^2 - \frac{29}{4}s + 4k + \frac{127}{2} > \frac{1}{4}(\frac{8}{9}k)^2 - \frac{29}{4}(\frac{8}{9}k) + 4k + \frac{127}{2} \\ &\geq \frac{1}{6}k^2 + \frac{225}{48}, \end{aligned}$$

and we are done.

If $k - s = 3$. Then $s = k - 3$. So

$$\begin{aligned} N &\geq 7 + (\frac{1}{4}s^2 - \frac{1}{4}s - \frac{1}{2}) - 3s + 65 = \frac{1}{4}s^2 - \frac{13}{4}s + \frac{143}{2} \\ &= \frac{1}{4}(k-3)^2 - \frac{13}{4}(k-3) + \frac{143}{2} \geq \frac{1}{6}k^2 + \frac{225}{48}, \end{aligned}$$

and we are done.

Next assume that $k - s \leq 2$. Then $s \geq k - 2$. Note that $2^{k-s} - 1 \geq k - s$. Then

$$\begin{aligned} N &\geq k - s + (\frac{1}{4}s^2 - \frac{1}{4}s - \frac{1}{2}) - 3s + 65 = \frac{1}{4}s^2 - \frac{17}{4}s + k + \frac{129}{2} \\ &> \frac{1}{4}(k-2)^2 - \frac{17}{4}(k-2) + k + \frac{129}{2} \\ &\geq \frac{1}{6}k^2 + \frac{225}{48}, \end{aligned}$$

and we are done.

Case 5. $u = 7$. Then $k \geq s > 5\left(\frac{3}{2}\right)^{u-3} + u - 2 > 30$ and $\Delta = \frac{1}{4}(2s-u+1)(u-2) - 5\left(\frac{3}{2}\right)^{u-2} + 2 < \frac{5}{2}s - 43$. Using the same argument as in Case 4, we have that $N \geq \frac{1}{6}k^2 + \frac{225}{48}$.

Case 6. $u = 6$. Then $\Delta = \frac{1}{4}(2s-u+1)(u-2) - 5\left(\frac{3}{2}\right)^{u-2} + 2 < 2s - 28$. Similar to Case 4, we obtain that $N \geq \frac{1}{6}k^2 + \frac{225}{48}$.

This completes the proof.

Lemma 5.3. *Let k, s, u, v be positive integers such that $3 \le v \le u+1 \le s \le k$, $k \ge 29$, $s > \frac{8}{9}k$, and $5\left(\frac{3}{2}\right)^{u-3} + u - 2 < s \le 5\left(\frac{3}{2}\right)^{u-2} + u - 1$. Then*

$$2^{k-s} - 1 + 5\left(\frac{3}{2}\right)^{v-3} - 1 + \frac{1}{3}(s-v)^2 \ge \frac{1}{6}k^2.$$

PROOF. Let $N = 2^{k-s} - 1 + 5\left(\frac{3}{2}\right)^{v-3} - 1 + \frac{1}{3}(s-v)^2$. Note that $s > \frac{8}{9}k$, $k \ge 29$, and $2^{k-s} - 1 \ge k - s \ge 0$.

Case 1. $v \le 5$. Then

$$N \ge \frac{1}{3}(s-v)^2 > \frac{1}{3}(\frac{8}{9}k - 5)^2 \ge \frac{1}{6}k^2,$$

and we are done.

Case 2. $v = 6$. Then $5\left(\frac{3}{2}\right)^{6-3} - 1 = \frac{127}{8}$. So

$$\begin{aligned} N &\ge k - s + \frac{127}{8} + \frac{1}{3}(s-6)^2 = \frac{1}{3}s^2 - 5s + \frac{223}{8} + k \\ &> \frac{1}{3}(\frac{8}{9}k)^2 - 5(\frac{8}{9}k) + \frac{223}{8} + k \ge \frac{1}{6}k^2, \end{aligned}$$

and we are done.

Case 3. $7 \le v \le 9$. Similar to Case 2, we obtain that $N \ge \frac{1}{6}k^2$.

Case 4. $v \ge 10$. Then $5\left(\frac{3}{2}\right)^{10-3} - 1 > 84$. Since $u \ge v - 1$, we infer that $u \ge 9$. Similar to Case 1 in the proof of Lemma 5.2, we can show that $s > u - 2 + \frac{10}{3}\left(\frac{3}{2}\right)^{u-2} > \frac{17}{2}(u-2)$. Therefore, $u - 2 < \frac{2}{17}s$ and $s - v \ge s - u - 1 \ge \frac{15}{17}s - 3$.

Note that $s \ge \frac{8}{9}k$. Then

$$\begin{aligned} N &\ge k - s + 84 + \frac{1}{3}(\frac{15}{17}s - 3)^2 \\ &= \frac{1}{3}(\frac{15}{17})^2 s^2 - \frac{47}{17}s + k + 87 \\ &> \frac{1}{3}(\frac{15}{17})^2(\frac{8}{9}k)^2 - \frac{47}{17}(\frac{8}{9}k) + k + 87 \ge \frac{1}{6}k^2, \end{aligned}$$

and we are done.

This completes the proof.

Now we are in the position to prove Theorem 1.1

Proof of Theorem 1.1. By Corollary 1.5, we have that $f(k) \geq \frac{1}{6}k^2$ holds for $1 \leq k \leq 28$. Next we assume that $k \geq 29$ and suppose that $f(m) \geq \frac{1}{6}m^2$ holds for every positive integers $m < k$.

Let S be a zero-sum free generating subset of a finite abelian group G with $|S| = k$. Write S as $S = S_1S_2$, where S_1 and S_2 are two disjoint subsets of S such that $\mathrm{ord}(x) = 2$ for every $x \in S_1$ and $\mathrm{ord}(y) \geq 3$ for every $y \in S_2$.

If $|S_1| \geq \frac{1}{2}k > 4$, we have that $2^{|S_1|} \geq |S_1|^2$. It follows from Lemma 2.4 that

$$|\Sigma(S)| \geq 2^{|S_1|}(k - |S_1| + 1) - 1 \geq |S_1|^2 - 1 \geq \frac{1}{4}k^2 - 1 > \frac{1}{6}k^2,$$

and we are done. If $\frac{1}{2}k > |S_1| \geq \frac{1}{9}k > 3$, we have $|S_1| \geq 4$ and thus $2^{|S_1|} \geq |S_1|^2$. It follows from Lemma 2.4 that

$$\begin{aligned} |\Sigma(S)| &\geq 2^{|S_1|}(k - |S_1| + 1) - 1 \geq |S_1|^2(k - |S_1|) - 1 \\ &\geq \frac{8}{729}k^3 - 1 > \frac{1}{6}k^2, \end{aligned}$$

and we are done. Next we may assume that $|S_1| < \frac{k}{9}$ and thus $|S_2| > \frac{8}{9}k \geq 25$.

Let $s = |S_2| \geq 25$. Note that $S_2 \cap (-S_2) = \emptyset$. Now applying Lemma 5.1 to S_2, we obtain that there exist a subset $T \subset S_2$ and integers $u, v, q \in [1, s]$ satisfying $3 \leq u \leq q \leq s$, $1 \leq v \leq q$, $|T| = s - v$, and

$$5\left(\frac{3}{2}\right)^{u-3} + u - 2 < s \leq 5\left(\frac{3}{2}\right)^{u-2} + u - 1,$$

such that

$$|\Sigma(S_2)| \geq \begin{cases} \frac{1}{4}(s+1)(s-2) - \Delta, & \text{if } q = s; \\ \Omega + 1 + 2|\Sigma(T)|, & \text{if } q < s. \end{cases}$$

where

$$\Delta = \frac{1}{4}(2s - u + 1)(u - 2) - 5\left(\frac{3}{2}\right)^{u-2} + 2,$$

and

$$\Omega = \begin{cases} 0, & \text{if } v = 1; \\ 1, & \text{if } v = 2; \\ 5\left(\frac{3}{2}\right)^{v-3} - 2, & \text{if } 2 \leq v - 1 \leq u; \\ \frac{1}{4}(2s - v + 2)(v - 3) - \Delta, & \text{if } v - 1 > u. \end{cases}$$

By the inductive assumption we have that $|\Sigma(T)| \geq \frac{1}{6}(s-v)^2$. By Lemma 2.4, we have that $|\Sigma(S_1)| = 2^{k-s} - 1$. It follows from Lemma 2.2 that

$$|\Sigma(S)| \geq |\Sigma(S_1)| + |\Sigma(S_2)|,$$

and therefore,

$$|\Sigma(S)| \geq \begin{cases} 2^{k-s} - 1 + \frac{1}{4}(s+1)(s-2) - \Delta, & \text{if } q = s; \\ 2^{k-s} - 1 + \Omega + 1 + \frac{1}{3}(s-v)^2, & \text{if } q < s. \end{cases}$$

We distinguish three cases according to the values of q, u, and v.

Case 1. $q = s$. Then $|\Sigma(S)| \geq 2^{k-s} - 1 + \frac{1}{4}(s+1)(s-2) - \Delta$. It follows from Lemma 5.2 that $|\Sigma(S)| \geq \frac{1}{6}k^2 + \frac{225}{48} > \frac{1}{6}k^2$, and we are done.

Case 2. $q < s$ and $v - 1 \leq u$. If $v = 1$ or $v = 2$, then

$$|\Sigma(S)| \geq \frac{1}{3}(s-2)^2 \geq \frac{1}{3}(\frac{8}{9}k - 2)^2 > \frac{1}{6}k^2.$$

Next we assume that $v \geq 3$. Then

$$|\Sigma(S)| \geq 2^{k-s} - 1 + 5\left(\frac{3}{2}\right)^{v-3} - 1 + \frac{1}{3}(s-v)^2.$$

It follows from Lemma 5.3 that $|\Sigma(S)| \geq \frac{1}{6}k^2$, and we are done.

Case 3. $q < s$ and $v - 1 > u$. Then

$$\begin{aligned} |\Sigma(S)| &\geq 2^{k-s} - 1 + \frac{1}{4}(2s - v + 2)(v-3) - \Delta + 1 + \frac{1}{3}(s-v)^2 \\ &= 2^{k-s} - 1 + \frac{1}{4}(s+1)(s-2) - \Delta + \frac{1}{12}(s - v - \frac{15}{2})^2 - \frac{225}{48}. \end{aligned}$$

It follows from Lemma 5.2 that

$$|\Sigma(S)| \geq \frac{1}{6}k^2 + \frac{225}{48} + \frac{1}{12}(s - v - \frac{15}{2})^2 - \frac{225}{48} \geq \frac{1}{6}k^2,$$

and we are done.

This completes the proof.

6. On the multiplicity of zero-sum free sequence

In this section, we estimate the multiplicity of an element in a zero-sum free sequence over finite cyclic groups. We will prove our last main result.

Proof of Theorem 1.6. Let $q \in \mathbb{N}_0$ be maximal such that S has a representation in the form $S = S_0 \cdot S_1 \cdot \ldots \cdot S_q$, where $S_1, \ldots, S_q$ are zero-sum free subsets of G with length $|S_\nu| = 14$ for all $\nu \in [1, q]$. Among all those representations of S choose one for which $d = |\mathrm{supp}(S_0)|$ is maximal, and set $S_0 = g_1^{r_1} \cdot \ldots \cdot g_d^{r_d}$, where $g_1, \ldots, g_d$ are pairwise distinct, $r_1 \geq \ldots \geq r_d \geq 1$ and $d \in [0, 13]$.

We first show that $r_1 \geq 2$. Assume to the contrary that $r_1 \leq 1$. Then $d = 0$ or $r_1 = \ldots = r_d = 1$. Let $f(0) = 0$. Then it follows from Lemma 2.2 and Theorem 1.4 that

$$\begin{aligned} |\Sigma(S)| &\geq |\Sigma(S_0)| + \sum_{i=1}^{q} |\Sigma(S_i)| \geq f(d) + 66q \\ &= f(d) + 66\frac{|S| - d}{14} \geq \frac{14f(d) + 66|S| - 66d}{14} \\ &\geq \frac{66|S| - 152}{14} \geq n, \end{aligned}$$

yielding a contradiction to that S is zero-sum free. Thus $r_1 \geq 2$. By the maximality of $|\mathrm{supp}(S_0)|$, we infer that $g_1 \in S_\mu$ for every $\mu \in [1, q]$. Otherwise, there exists $j \in [1, q]$ such that $g_1 \notin S_j$, say $g_1 \notin S_1$. Then there exists $h \in S_1$ such that $h \notin \mathrm{supp}(S_0)$. Hence S allows a representation in the form $S = (S_0 g_1^{-1} h) \cdot (S_1 h^{-1} g_1) \cdot S_2 \cdot \ldots \cdot S_q$ and $|\mathrm{supp}(S_0 g_1^{-1} h)| > |\mathrm{supp}(S_0)|$, yielding a contradiction.

Set $g = g_1$. Next we can write S_0 as

$$S_0 = \prod_{i=1}^{13} T_1^{(i)} \cdot \ldots \cdot T_{q_i}^{(i)},$$

where $q_i \in \mathbb{N}_0$ for all $i \in [1, 13]$, $T_\nu^{(i)}$ is a zero-sum free subset of G with $\mathsf{v}_g(T_\nu^{(i)}) = 1$ and $|T_\nu^{(i)}| = i$ for all $\nu \in [1, q_i]$. Thus we have

$$|S| = 14q + |S_0| = 14q + \textstyle\sum_{i=1}^{13} iq_i \text{ and } \mathsf{v}_g(S) = q + \textstyle\sum_{i=1}^{13} q_i.$$

Since S is zero-sum free, it follows from Lemma 2.2 and Theorem 1.4 that

$$\begin{aligned} n-1 &\geq |\Sigma(S)| \geq |\Sigma(S_0)| + \sum_{i=1}^{q} |\Sigma(S_i)| \\ &\geq \sum_{i=1}^{13}\sum_{j=1}^{q_i} |\Sigma(T_j^{(i)})| + \sum_{i=1}^{q} |\Sigma(S_i)| \\ &\geq \sum_{i=1}^{13} q_i f(i) + 66q. \end{aligned}$$

We infer that

$$\begin{aligned} & 7|S| - (n-1) \\ \leq\ & 7(14q + \sum_{i=1}^{13} iq_i) - (\sum_{i=1}^{13} q_i f(i) + 66q) \\ \leq\ & 32q + 30q_{13} + 32q_{12} + 30q_{11} + 29q_{10} + 28q_9 + 26q_8 + \\ & \quad 25q_7 + 23q_6 + 22q_5 + 20q_4 + 16q_3 + 11q_2 + 6q_1 \\ \leq\ & 32\mathsf{v}_g(S). \end{aligned}$$

Therefore, $\mathsf{v}_g(S) \geq \frac{7|S|-n+1}{32}$.

This completes the proof.

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