ON THE LOWER BOUNDS OF DAVENPORT CONSTANT

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ABSTRACT. Let $G = C_{n_1} \oplus \cdots \oplus C_{n_r}$ with $1 < n_1 | \ldots | n_r$ be a finite abelian group. The Davenport constant $\mathsf{D}(G)$ is the smallest integer t such that every sequence S over G of length $|S| \geq t$ has a non-empty zero-sum subsequence. It is a starting point of zero-sum theory. It has a trivial lower bound $\mathsf{D}^*(G) = n_1 + \cdots + n_r - r + 1$, which equals $\mathsf{D}(G)$ over p-groups. We investigate the non-dispersive sequences over groups C_n^r, thereby revealing the growth of $\mathsf{D}(G) - \mathsf{D}^*(G)$ over non-p-groups $G = C_n^r \oplus C_{kn}$ with $n, k \neq 1$. We give a general lower bound of $\mathsf{D}(G)$ over non-p-groups and show that if G is an abelian group with $\exp(G) = m$ and rank r, fix $m > 0$ a non-prime-power, then for each $N > 0$ there exists an $\varepsilon > 0$ such that if $|G|/m^r < \varepsilon$, then $\mathsf{D}(G) - \mathsf{D}^*(G) > N$.

1. Introduction and main results

The Davenport constant has been studied since the 1960s. It naturally occurs in various branches of combinatorics, number theory, and geometry (see [10, Chapter 5] and [7]). Early work on the Davenport constant and on the Erdős-Ginzburg-Ziv Theorem are considered as starting points of zero-sum theory. The goal of the present paper is to provide new lower bounds for the Davenport constant.

Let G be an additively written finite abelian group, say $G \cong C_{n_1} \oplus \cdots \oplus C_{n_r}$, where $r = \mathsf{r}(G)$ is the rank of G and $1 < n_1 | \ldots | n_r$, and set $\mathsf{D}^*(G) = 1 + \sum_{i=1}^r (n_i - 1)$. If $S = g_1 \cdot \ldots \cdot g_\ell$ is a sequence over G, then $|S| = \ell$ is its length and S is called a zero-sum sequence if its sum $\sigma(S) = g_1 + \cdots + g_\ell$ is equal to 0. The Davenport constant $\mathsf{D}(G)$ of G is the smallest integer $\ell \in \mathbb{N}$ such that every sequence S over G of length $|S| \geq \ell$ has a nonempty zero-sum subsequence. A straightforward example shows that $\mathsf{D}^*(G) \leq \mathsf{D}(G)$. Already in the 1960s it was proved that equality holds for p-groups and for groups having rank $r(G) \leq 2$ (see [10, Chapter 5]). Here we refer to a couple of papers ([1, 2, 13, 15, 16], [9, Corollary 4.2.13]) of the last decade offering a growing list of groups G satisfying $\mathsf{D}(G) = \mathsf{D}^*(G)$. However, it is still open whether or not equality holds for groups of rank three or for groups of the form C_n^r.

The first example of $\mathsf{D}(G) > \mathsf{D}^*(G)$ is due to P.C. Baayen in 1969. Let $G = C_2^{4k} \oplus C_{4k+2}$ with $k \in \mathbb{N}_+$, then $\mathsf{D}(G) \geq \mathsf{D}^*(G) + 1$ ([3, Theorem 8.1]). We briefly introduce some works on the lower bounds of Davenport constant.

(1) Let $G = C_n^{(k-1)n+\rho} \oplus C_{kn}$ with $n, k \geq 2$, $\gcd(n, k) = 1$ and $0 \leq \rho \leq n - 1$.
   (a) If $\rho \geq 1$ and $\rho \not\equiv n \pmod k$, then $\mathsf{D}(G) \geq \mathsf{D}^*(G) + \rho$.
   (b) If $\rho \leq n - 2$ and $x(n - \rho + 1) \not\equiv n \pmod k$ for any $x \in [1, n-1]$, then $\mathsf{D}(G) \geq \mathsf{D}^*(G) + \rho + 1$. (Emde Boas and Kruyswijk, 1969)

(2) Let $G = C_m \oplus C_n^2 \oplus C_{2n}$ with $m, n \in \mathbb{N}_{\geq 3}$ odd and $m | n$. Then $\mathsf{D}(G) \geq \mathsf{D}^*(G) + 1$. (Geroldinger and Schneider, [12], 1992)

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(3) Let $G = C_2^{r-1} \oplus C_{2k}$ with $k > 1$ odd. Then $\mathsf{D}(G) - \mathsf{D}^*(G) \geq \max\{\log_2 r - \alpha(k) - 2k + 1, 0\}$, where $\alpha(k) = i$ iff $2^{i-1} + 1 < k \leq 2^i + 1$. (Mazur, [19], 1992)

(4) Let $G = C_2^i \oplus C_{2n}^{5-i}$ with $i \in [1, 4]$ and $n \geq 3$ odd. Then $\mathsf{D}(G) \geq \mathsf{D}^*(G) + 1$. (See [6, 11, 12] for $i = 2$, $i = 1$ and $i \in \{3, 4\}$ separately)

The third result shows the growth of $\mathsf{D}(G) - \mathsf{D}^*(G)$ over $G = C_2^{r-1} \oplus C_{2k}$ with k odd. The author, Mazur, also asked if there are similar results when k is even [19].

This paper will show the growth of $\mathsf{D}(G) - \mathsf{D}^*(G)$ over non-p-groups $G = C_n^r \oplus C_{kn}$ with any $n, k \neq 1$ (see Theorem 4.3 and Corollary 4.4). We show $\mathsf{D}(G) - \mathsf{D}^*(G)$ grows at least logarithmically with respect to r for a fixed n. For the cases of $\gcd(k, n) \neq 1$, this is the first time to prove $\mathsf{D}(G) = \mathsf{D}^*(G)$ false. We show that $\mathsf{D}(G) - \mathsf{D}^*(G) > 0$ can happen even if the exponent of G is arbitrarily large (see Remark 4.5). So Mazur's result is improved and more results are derived.

We prove the result with a new method. By Lemma 4.1, this paper connects the lower bounds of Davenport constant to the study of *non-dispersive sequence*, which goes back to a conjecture of Graham reported in [4]. A sequence S over G is called non-dispersive if all nonempty zero-sum subsequences of S have the same length. In 1976, Erdős and Szemerédi [4] proved that if S is a non-dispersive sequence over C_p of length p, then S takes at most two distinct values, where p is a sufficiently large prime. Gao *et al.* [5] and Grynkiewicz [17] independently improved this result to all positive integers. A related question was naturally proposed by Girard [14] to determine the longest length of non-dispersive sequences over any group G. The answer is known for group C_2^r (see [8]). We investigate non-dispersive sequences over groups C_n^r with $n \geq 2$ (see Theorem 3.1), thereby improving the lower bounds of Davenport constant over $C_n^r \oplus C_{kn}$.

We also give general lower bounds for all non-p-groups (see Theorem 4.6) and some other interesting corollaries.

2. Preliminaries

Our notation and terminology of sequences over abelian groups is consistent with [10, 18].

Let $n \in \mathbb{N}_{\geq 2}$ and set $n = pq$ for some prime p and some $q \in [1, n]$. For any $\ell \in \mathbb{N}_+$, we define $\theta(\ell; p)$, $\omega(\ell; n, p)$ and $\mathbf{M}(\ell; p, q)$ as follows through out this paper.

1.
$$\theta(\ell; p) = \begin{cases} \frac{2(p^\ell - 1)}{p-1} - \ell, & \text{if } p > 2 \\ 2^\ell - 1 - \ell, & \text{if } p = 2 \end{cases}.$$

2.
$$\omega(\ell; n, p) = \begin{cases} p^{\ell-1} n, & \text{if } p > 2 \\ 2^{\ell-2} n, & \text{if } p = 2 \end{cases}.$$

3. For any $\ell \in \mathbb{N}_+$, the set $\mathbf{M}(\ell; p, q)$ is constructed by a recursive algorithm:

(i) $\mathbf{M}(1; p, q) = \begin{cases} \{q, (p-1)q\}, & \text{if } p > 2 \\ \{q\}, & \text{if } p = 2 \end{cases}.$

(ii) $\mathbf{M}(\ell+1; p, q) = \mathbf{M}(\ell; p, q) \times \mathbf{A} \cup \{0\}^\ell \times \mathbf{M}(1; p, q)$, where $\mathbf{A} = \{0, q, \dots, (p-1)q\}$.

Remark 2.1. *We can use a direct way to construct $\mathbf{M}(\ell)$ for $\ell \in \mathbb{N}_+$, apart from the recursive algorithm given before. In* (4)*, we can let $a_s = 1$ and derive that*

$$\mathbf{M}(\ell) = \bigcup_{t=0}^{\ell-1} \{0\}^t \times \mathbf{M}(1) \times \mathbf{A}^{\ell-t-1}.$$

Hence it follows a direct way to construct the non-dispersive sequences (in Theorem 3.1) and zero-sum free sequences with the techniques in Lemma 4.1.

Let $|w|_n$ denote the least nonnegative residue of an integer w modulo n. Let $|\mathbf{B}|$ denote the cardinality of a set $\mathbf{B}$.

The elements of $\mathbf{M}(\ell;p,q)$ are ℓ-tuples of integers. We list the elements of $\mathbf{M}(\ell;p,q)$ in some fixed but arbitrary order. Then $\mathbf{M}(\ell;p,q)[i,j]$ denotes the i-th entry of the j-th element of $\mathbf{M}(\ell;p,q)$. We often fix some n,p and q before considering $\theta(\ell;p)$, $\omega(\ell;n,p)$ and $\mathbf{M}(\ell;p,q)$. For convenience, we might omit the parameters "n", "p" and "q" when no misunderstanding is likely to occur. Thus, $\theta(\ell)$, $\omega(\ell)$ and $\mathbf{M}(\ell)[i,j]$ will mean $\theta(\ell;p)$, $\omega(\ell;n,p)$ and $\mathbf{M}(\ell;p,q)[i,j]$ unless otherwise stated.

Proposition 2.2. *Let $n \in \mathbb{N}_{\geq 2}$ and set $n = pq$ for some prime p and some $q \in [1,n]$. For any $\ell \in \mathbb{N}_+$, $\mathbf{M}(\ell;p,q)$ has following three properties:*

i. $|\mathbf{M}(\ell)| = \theta(\ell) + \ell$.

ii. For any $1 \leq a_1 < \cdots < a_s \leq \ell$ and any $v_i \in [1, p-1]$ with $a_i, v_i \in \mathbb{N}_+$ and $i \in [1,s]$, we have

$$\sum_{j=1}^{|\mathbf{M}(\ell)|} \Big|\sum_{i=1}^{s} v_i \mathbf{M}(\ell)(a_i,j)\Big|_n = \omega(\ell).$$

iii.

$$\sum_{j=1}^{|\mathbf{M}(\ell)|} \Big|-\sum_{i=1}^{s} v_i \mathbf{M}(\ell)[a_i,j]\Big|_n = \sum_{j=1}^{|\mathbf{M}(\ell)|} \Big|\sum_{i=1}^{s} v_i \mathbf{M}(\ell)[a_i,j]\Big|_n.$$

Proof.

1) By the definition of $\mathbf{M}(\ell)$ we can derive that $|\mathbf{M}(\ell+1)| = |\mathbf{M}(\ell)|\cdot p + |\mathbf{M}(1)|$, thus $|\mathbf{M}(\ell)| = \frac{(p^\ell-1)|\mathbf{M}(1)|}{p-1} = \theta(\ell)+\ell$.

2) **Case 1.** $\ell = 1$.

In this case, $s = 1$ and $a_1 = 1$. By the definitions of $\mathbf{M}(1)$ and v_1, it is easy to infer that

$$\sum_{j=1}^{|\mathbf{M}(1)|} \big|v_1 \mathbf{M}(1)[1,j]\big|_n = \begin{cases} pq, & \text{if } p > 2 \\ q, & \text{if } p = 2 \end{cases}.$$

Case 2. $\ell \geq 2$ and $a_s = \ell$.

By the rules of Cartesian product and the definition of $\mathbf{M}(1)$, we derive that

$$\begin{aligned}\mathbf{M}(\ell) &= \mathbf{M}(\ell-1)\times \mathbf{A} \cup \{0\}^{\ell-1}\times \mathbf{M}(1)\\ &= (\bigcup_{t=0}^{p-1} \mathbf{M}(\ell-1)\times\{tq\}) \cup \{0\}^{\ell-1}\times\mathbf{M}(1).\end{aligned}$$

Consequently,

$$\begin{aligned}\sum_{j=1}^{|\mathbf{M}(\ell)|} \Big|\sum_{i=1}^{s} v_i \mathbf{M}(\ell)[a_i,j]\Big|_n =& \sum_{t=0}^{p-1}\sum_{j=1}^{|\mathbf{M}(\ell-1)|} \Big|\sum_{i=1}^{s-1} v_i \mathbf{M}(\ell-1)[a_i,j] + v_s tq\Big|_n \\ &+ \sum_{j=1}^{|\mathbf{M}_1|} \big|0 + v_s \mathbf{M}(1)[1,j]\big|_n.\end{aligned} \tag{1}$$

Note that, for any $x \in \{0, q, \dots, (p-1)q\}$, by $v_s \in [1, p-1]$, we have $\gcd(v_s, p) = 1$. Thus

$$\sum_{t=0}^{p-1} |x + v_s tq|_n = 0 + q + \cdots + (p-1)q = \frac{(p-1)pq}{2}. \tag{2}$$

Every $\mathbf{M}(\ell-1)[a_i, j]$ is in $\{0, q, \dots, (p-1)q\}$. Therefore

$$\sum_{i=1}^{s-1} v_i \mathbf{M}(\ell-1)[a_i, j] \in \{0, q, \dots, (p-1)q\}. \tag{3}$$

By (1), (2) and (3), we have

$$\begin{aligned}\sum_{j=1}^{|\mathbf{M}(\ell)|} \Big|\sum_{i=1}^{s} v_i \mathbf{M}(\ell)[a_i, j]\Big|_n &= \sum_{j=1}^{|\mathbf{M}(\ell-1)|} \frac{(p-1)pq}{2} + \sum_{j=1}^{|\mathbf{M}(1)|} \big|v_s \mathbf{M}(1)[1, j]\big|_n \\ &= \frac{(p^{\ell-1}-1)|\mathbf{M}(1)|}{p-1} \cdot \frac{(p-1)pq}{2} + \omega(1) \\ &= \begin{cases} p^{\ell} q, & \text{if } p > 2 \\ 2^{\ell-1} q, & \text{if } p = 2 \end{cases}.\end{aligned}$$

Case 3. $\ell \geq 2$ and $a_s < \ell$.

Indeed, by the definition of $\mathbf{M}_\ell$ and the rules of Cartesian product, we have

$$\begin{aligned}\mathbf{M}(\ell) &= \mathbf{M}(\ell-1) \times \mathbf{A} \cup \{0\}^{\ell-1} \times \mathbf{M}(1) \\ &= (\mathbf{M}(\ell-2) \times \mathbf{A} \cup \{0\}^{\ell-2} \times \mathbf{M}(1)) \times \mathbf{A} \cup \{0\}^{\ell-1} \times \mathbf{M}(1) \\ &= \mathbf{M}(\ell-2) \times \mathbf{A}^2 \cup \{0\}^{\ell-2} \times \mathbf{M}(1) \times \mathbf{A} \cup \{0\}^{\ell-1} \times \mathbf{M}(1) \\ &\vdots \\ &= \mathbf{M}(a_s) \times \mathbf{A}^{\ell-a_s} \bigcup_{t=a_s}^{\ell-1} \{0\}^t \times \mathbf{M}(1) \times \mathbf{A}^{\ell-t-1}.\end{aligned} \tag{4}$$

Thus by (4) and $|\mathbf{A}^{\ell-a_s}| = p^{\ell-a_s}$, together with the result in **Case 2.**, we can derive that

$$\begin{aligned}\sum_{j=1}^{|\mathbf{M}(\ell)|} \Big|\sum_{i=1}^{s} v_i \mathbf{M}(\ell)[a_i, j]\Big|_n &= \sum_{j=1}^{|\mathbf{M}(a_s)|} \Big|\sum_{i=1}^{s} v_i \mathbf{M}(a_s)[a_i, j]\Big|_n \cdot p^{\ell-a_s} + 0 \\ &= \omega(a_s) \cdot p^{\ell-a_s} = \begin{cases} p^{\ell} q, & \text{if } p > 2 \\ 2^{\ell-1} q, & \text{if } p = 2 \end{cases}.\end{aligned}$$

3) Since $\mathbf{A} = -\mathbf{A}$ and $\mathbf{M}(1) = -\mathbf{M}(1)$, by the definition of $\mathbf{M}(\ell)$, it follows that $\mathbf{M}(\ell) = -\mathbf{M}(\ell)$. Thus it is easy to infer that

$$\sum_{j=1}^{|\mathbf{M}(\ell)|} \Big|-\sum_{i=1}^{s} v_i \mathbf{M}(\ell)[a_i, j]\Big|_n = \sum_{j=1}^{|\mathbf{M}(\ell)|} \Big|\sum_{i=1}^{s} v_i \mathbf{M}(\ell)[a_i, j]\Big|_n.$$

□

We need the following result which is a straightforward consequence of [12, Lemma 1] and we omit the similar proof here.

Lemma 2.3. *Let* $G = C_{n_1} \oplus C_{n_2} \oplus \cdots \oplus C_{n_r}$ *with* $1 < n_1 | n_2 \dots | n_r$. *Let* $H_x = \oplus_{i \in I_x} C_{n_i}$, *where* $x \in [1, z]$, $z \in \mathbb{N}_+$, $\varnothing \neq I_x \subsetneq [1, r]$ *and* $I_x \cap I_y = \varnothing$ *for any*

$x, y \in [1, z]$. *Then*

$$\mathsf{D}(G) - \mathsf{D}^*(G) \geq \sum_{x=1}^{z} (\mathsf{D}(H_x) - \mathsf{D}^*(H_x)).$$

3. On non-dispersive sequences over C_n^r

In this section, we will construct long non-dispersive sequences by $\mathbf{M}(\ell)$'s.

Theorem 3.1. *Let $G = C_n^r$, where $r \in \mathbb{N}_+$ and $n \in \mathbb{N}_{\geq 2}$, and let p be a prime divisor of n. If $\ell \in \mathbb{N}_+$ such that $r \geq \theta(\ell; p) \geq 1$, then there exists a sequence S over G of length*

$$|S| = (n-1)r + (p-1)\ell = \mathsf{D}^*(G) + (p-1)\ell - 1,$$

such that every nonempty zero-sum subsequence T of S is length of

$$|T| = \omega(\ell; n, p).$$

Proof. Set $n = pq$, where $q \in [1, n]$.
Case 1. $p > 2$.

It follows from $r \geq \theta(\ell; p) \geq 1$ that $\ell \geq 1$. Let

$$\mathbf{E}(\ell) = \bigcup_{t=0}^{\ell-1} \{0\}^t \times \{q, (p-1)q\} \times \{0\}^{\ell-t-1}$$

and

$$\mathbf{F}(\ell) = \bigcup_{t=0}^{\ell-1} \{0\}^t \times \{1\} \times \{0\}^{\ell-t-1}.$$

Let $\mathbf{W}(\ell) = \mathbf{M}(\ell) \backslash \mathbf{E}(\ell) \cup \mathbf{F}(\ell)$. Thus by Proposition 2.2, we have $|\mathbf{W}(\ell)| = |\mathbf{M}(\ell)| - 2\ell + \ell = \theta(\ell)$.

List the elements of $\mathbf{W}(\ell)$ in some fixed but arbitrary order. Let $\mathbf{W}(\ell)[i, j]$ denote the i-th entry of the j-th element of $\mathbf{W}(\ell)$. For any indices

$$1 \leq a_1 < \cdots < a_s \leq \ell \text{ and } v_i \in [1, p-1] \text{ with } i \in [1, s],$$

also by Proposition 2.2 and $n = pq$, we have

$$\begin{aligned} &\sum_{j=1}^{|\mathbf{W}(\ell)|} \Big|\sum_{i=1}^{s} v_i \mathbf{W}(\ell)[a_i, j]\Big|_n \\ =& \sum_{j=1}^{|\mathbf{M}(\ell)|} \Big|\sum_{i=1}^{s} v_i \mathbf{M}(\ell)[a_i, j]\Big|_n - \sum_{i=1}^{s} (|v_i q|_n + |v_i(p-1)q|_n) + \sum_{i=1}^{s} |v_i|_n \\ =& \omega(\ell) - \sum_{i=1}^{s} n + \sum_{i=1}^{s} (n - v_i) = \omega(\ell) - \sum_{i=1}^{s} v_i. \end{aligned} \tag{5}$$

Let $C_n^r = \oplus_{j=1}^r \langle e_j \rangle$ with $\mathrm{ord}(e_j) = n$ for each $j \in [1, r]$. By $r \geq \theta(\ell; p)$, we can set

$$x_b = \sum_{j=1}^{\theta(\ell)} \mathbf{W}(\ell)[b, j] \cdot e_j, \text{ where } b \in [1, \ell],$$

and let sequence

$$S = \prod_{j=1}^{r} e_j^{n-1} \prod_{b=1}^{\ell} x_b^{p-1}.$$

Suppose that S_1 is a nonempty zero-sum subsequence of S. If x_b does not occur in S_1 for any $b \in [1,\ell]$, then S_1 is zero-sum free. Thus, for any indices $1 \le a_1 < \cdots < a_s \le \ell$ and any $v_i \in [1, p-1]$ with $i \in [1,s]$, we set

$$S_1 = \prod_{j=1}^{r} e_j^{u_j} \prod_{i=1}^{s} x_{a_i}^{v_i},$$

where $u_j \in [0, n-1]$. Since S_1 is zero-sum, we have

$$u_j = \big|n - \sum_{i=1}^{s} v_i \mathbf{W}(\ell)[a_i, j]\big|_n,\ j \in [1, \theta(\ell)],$$

and $u_j = 0$ for $j > \theta(\ell)$. Thus, together with (5) and Proposition 2.2, we obtain that

$$|S_1| = \sum_{i=1}^{s} v_i + \sum_{j=1}^{|\mathbf{W}(\ell)|} \big|n - \sum_{i=1}^{s} v_i \mathbf{W}(\ell)[a_i, j]\big|_n = \omega(\ell),$$

which completes the proof of this lemma in Case 1.

Case 2. $p = 2$.

It follows from $r \ge \theta(\ell; 2) \ge 1$ that $\ell \ge 2$.

Suppose that $r \ge 4$ and thus $\ell \ge 3$. Let

$$\mathbf{E}(\ell) = \bigcup_{t=0}^{\ell-1} \{0\}^t \times \{q\} \times \{0\}^{\ell-t-1},$$

$$\mathbf{F}(\ell) = \bigcup_{t=0}^{\ell-2} \{0\}^t \times \{q\} \times \{q\} \times \{0\}^{\ell-t-2},$$

$$\mathbf{H}(\ell) = \bigcup_{t=0}^{\ell-2} \{0\}^t \times \{1\} \times \{q\} \times \{0\}^{\ell-t-2},$$

$$\mathbf{I}(\ell) = \{q\} \times \{0\}^{\ell-2} \times \{q\}$$

and

$$\mathbf{J}(\ell) = \{q\} \times \{0\}^{\ell-2} \times \{1\}.$$

Let

$$\mathbf{W}(\ell) = \mathbf{M}(\ell) \backslash \mathbf{E}(\ell) \backslash \mathbf{F}(\ell) \cup \mathbf{H}(\ell) \backslash \mathbf{I}(\ell) \cup \mathbf{J}(\ell). \tag{6}$$

Thus by Proposition 2.2, we have $|\mathbf{W}(\ell)| = |\mathbf{M}(\ell)| - \ell - (\ell-1) + (\ell-1) - 1 + 1 = \theta(\ell)$. Let

$$\mathbf{U}(\ell) = \mathbf{M}(\ell) \backslash \big(\bigcup_{t=0}^{\ell-1} \{0\}^t \times \{q\} \times \{0\}^{\ell-t-1} \big).$$

By (6), for each $z \in [1,\ell]$, we can just change exactly one element $\mathbf{U}(\ell)[z, j_z]$ of $\mathbf{U}(\ell)$ from q to 1, to obtain $\mathbf{W}(\ell)$. Also it should satisfy that, for all $\mathbf{W}(\ell)[x, j_z]$ with $z \ne x \in [1,\ell]$, there exists exactly one element q and the others are 0, and if $z_1 \ne z_2$, then $j_{z_1} \ne j_{z_2}$, where $z_1, z_2 \in [1,\ell]$.

Hence, let indices $1 \le a_1 < \cdots < a_s \le \ell$, for any $z \in \{a_1, \ldots, a_s\}$, then either

$$\sum_{i=1}^{s} \mathbf{U}(\ell)[a_i, j_z] = q \text{ and } \sum_{i=1}^{s} \mathbf{W}(\ell)[a_i, j_z] = 1,$$

or

$$\sum_{i=1}^{s} \mathbf{U}(\ell)[a_i, j_z] = 2q \text{ and } \sum_{i=1}^{s} \mathbf{W}(\ell)[a_i, j_z] = q + 1.$$

So we have

$$\Big| -\sum_{i=1}^{s}\mathbf{W}(\ell)[a_i,j_z]\Big|_n - \Big| -\sum_{i=1}^{s}\mathbf{U}(\ell)[a_i,j_z]\Big|_n = q-1.$$

Together with Proposition 2.2 and $n=2q$, we have

$$\begin{aligned}
&\sum_{j=1}^{|\mathbf{W}(\ell)|}\Big|n-\sum_{i=1}^{s}\mathbf{W}(\ell)[a_i,j]\Big|_n\\
&=\sum_{j=1}^{|\mathbf{W}(\ell)|}\Big|-\sum_{i=1}^{s}\mathbf{W}(\ell)[a_i,j]\Big|_n\\
&=\sum_{j=1}^{|\mathbf{M}(\ell)|}\Big|-\sum_{i=1}^{s}\mathbf{M}(\ell)(a_i,j)\Big|_n-\sum_{i=1}^{s}|-q|_n+\sum_{z\in\{a_1,\dots,a_s\}}(q-1)\\
&=\sum_{j=1}^{|\mathbf{M}(\ell)|}\Big|\sum_{i=1}^{s}\mathbf{M}(\ell)[a_i,j]\Big|_n-sq+s(q-1)=\omega(\ell)-s.
\end{aligned}\tag{7}$$

Suppose that $\ell=2$, let $\mathbf{W}(2)=\{(1,1)\}$. It is clear that $|\mathbf{W}(2)|=\theta(2)=1$, and $\sum_{j=1}^{|\mathbf{W}(2)|}\big|n-\sum_{i=1}^{s}\mathbf{W}(2)[a_i,j]\big|_n=\omega(2)-s$ for any indices $1\le a_1<\cdots<a_s\le\ell$. Then by the similar proof in Case 1, we complete the proof. □

Definition 3.2. *([8]) Define* $\mathsf{disc}(G)$ *to be the smallest positive integer* t*, such that every sequence over* G *of length at least* t *has two nonempty zero-sum subsequences of distinct lengths.*

By Theorem 3.1, we can derive the following corollary immediately.

Corollary 3.3. *Let* $G=C_n^r$*, where* $r\in\mathbb{N}_+$ *and* $n\in\mathbb{N}_{\ge2}$*, and let* p *be a prime divisor of* n*. If* $\ell\in\mathbb{N}_+$ *such that* $r\in[\theta(\ell),\theta(\ell+1))$*, then* $\mathsf{disc}(G)\ge(n-1)r+(p-1)\ell+1$*.*

Note that, for $n=2$, the above bound equals $\mathsf{disc}(G)$ (see [8, Theorem 1.3]).

4. On the lower bounds of $\mathsf{D}(G)$

By Lemma 4.1 we connect the lower bounds of $\mathsf{D}(G)$ to special non-dispersive sequences. This lemma is a crucial one to this paper.

Lemma 4.1. *Let* $G=G_1\oplus\cdots\oplus G_t\oplus C_m$*, where* $t\in\mathbb{N}_+$*,* $m\in\mathbb{N}_{\ge2}$*, and* $G_1,\dots,G_t$ *are finite abelian groups. For every* $i\in[1,t]$*, let* S_i *be a non-dispersive sequence over* G_i *which only contains zero-sum subsequences of length* x_i*. If* $y=\sum_{i=1}^{t}\gcd(x_i,m)<m$*, then* $\mathsf{D}(G)\ge\sum_{i=1}^{t}|S_i|+m-y$*.*

Proof. By results from the elementary number theory, for every x_i with $i\in[1,t]$, there exists a $u_i\in[1,m-1]$ such that $|x_iu_i|_m=\gcd(x_i,m)$. Let $C_m=\langle e\rangle$. Consider the following sequence

$$S=(S_1+u_1e)(S_2+u_2e)\dots(S_t+u_te)e^{m-y-1}.$$

Suppose that S has a non-empty zero-sum subsequence T, and

$$T=T_1T_2\dots T_te^z \text{ with } T_i|(S_i+u_ie),\ i\in[1,t] \text{ and } 0\le z\le m-y-1.$$

We observe that the S_i's and e are independent and S_i only contains zero-sum subsequences of length x_i. Thus $|T_i|=x_i$ or $|T_i|=0$, for $i\in[1,t]$. And the sum of T is ve, where

$$v=\big||T_1|u_1+|T_2|u_2+\cdots+|T_t|u_t+z\big|_m.$$

Since T is non-empty and

$$\begin{aligned}&|x_1u_1|_m + |x_2u_2|_m + \cdots + |x_tu_t|_m + z\\ &= \sum_{i=1}^{t} \gcd(x_i, m) + z = y + z \le m - 1,\end{aligned}$$

it follows that $0 < v < m$ and thus T is not zero-sum. This contradicts the definition of T. Thus S is zero-sum free and $\mathsf{D}(G) \ge |S| + 1 = \sum_{i=1}^{t} |S_i| + m - y$. □

By Lemma 4.1, Theorem 3.1 and Lemma 2.3, we are able to construct long zero-sum free sequences over general abelian groups. Next, we would like to provide Theorem 4.3 and Corollary 4.4 to easily estimate the growth of $\mathsf{D}(G) - \mathsf{D}^*(G)$ for large r and $\exp(G)$.

Remark 4.2. *Let $G = C_n^r \oplus C_{kn}$ be a non-p-group with $n, k \in \mathbb{N}_{\ge 2}$. Then there exist p and k_1 such that p be a prime divisor of n, $k_1 \in \mathbb{N}_{\ge 2}$ be a divisor of k with* $\gcd(p, k_1) = 1$. *We use this remark to guarantee that the result in Theorem 4.3 is not vacuous.*

Proof. If $n = p^t > 1$ is a prime power, since G is a non-p-group, there exists $1 < k_1 | k$ with $\gcd(p, k_1) = 1$. If n has at least two distinct prime factors p_1 and p_2. Consider a prime factor p_3 of k, then either $\gcd(p_1, p_3) = 1$ or $\gcd(p_2, p_3) = 1$. Thus the existence is proved.

□

Theorem 4.3. *Let $G = C_n^r \oplus C_{kn}$ be a non-p-group with $n, k \in \mathbb{N}_{\ge 2}$. Let p be a prime divisor of n, $k_1 \in \mathbb{N}_{\ge 2}$ be a divisor of k, with $\gcd(p, k_1) = 1$, and $kn = k_1 m$ for some $m \in \mathbb{N}$. If $\ell \in \mathbb{N}_+$ and $t \in [1, k_1 - 1]$ with $r \ge t\theta(\ell) \ge 1$, then*

$$\mathsf{D}(G) \ge \mathsf{D}^*(G) + t(p-1)\ell - tm.$$

Proof. Let $(e_1, \ldots, e_r)$ be a basis of C_n^r with $\mathrm{ord}(e_1) = \cdots = \mathrm{ord}(e_r) = n$. Let

$$G_j = \oplus_{i=1+(j-1)\theta(\ell)}^{j\theta(\ell)} \langle e_i \rangle, \text{ where } j \in [1, t-1],$$

and let $G_t = \oplus_{i=1+(t-1)\theta(\ell)}^{r} \langle e_i \rangle$. By Theorem 3.1, there exists a sequence S_j over each G_j with

$$|S_j| = \mathsf{D}^*(G_j) - 1 + (p-1)\ell,$$

which only contains zero-sum subsequences of a unique length $\omega(\ell)$. Hence, by $\gcd(p, k_1) = 1$, we have

$$\begin{aligned}\gcd(\omega(\ell), kn) &\le \gcd(p^{\ell-1}n, kn) = n\gcd(p^{\ell-1}, k)\\ &= n\gcd\left(p^{\ell-1}, \frac{k}{k_1}\right) \le \frac{nk}{k_1} = m.\end{aligned}$$

And $\sum_{j=1}^{t} \gcd(\omega(\ell), kn) = tm < kn$. By Lemma 4.1, it follows that

$$\begin{aligned}\mathsf{D}(G) &\ge \sum_{j=1}^{t} |S_j| + kn - \sum_{j=1}^{t} \gcd(\omega(\ell), kn)\\ &\ge \sum_{j=1}^{t} |S_j| + kn - mt = \mathsf{D}^*(G) + ((p-1)\ell - m)t.\end{aligned}$$

□

Corollary 4.4. *Let $G = C_n^r \oplus C_{kn}$ be a non-p-group with $n, k \in \mathbb{N}_{\geq 2}$. Let p be a prime divisor of n, $k_1 \in \mathbb{N}_{\geq 2}$ be a divisor of k, with $\gcd(p, k_1) = 1$, and $kn = k_1 m$ for some $m \in \mathbb{N}$. For any integer $t \in [1, k_1 - 1]$, we have*

$$\mathsf{D}(G) > \mathsf{D}^*(G) + \frac{t(p-1)}{\log p} \log r - t(p-1)(\log_p t + 1) - tm. \tag{8}$$

Proof. In Remark 4.2, we proved the existence of p and k_1. For every $r \in \mathbb{N}_+$, there exists an $\ell \in \mathbb{N}_+$ such that $\theta(\ell) \geq 1$ and $r \in [t\theta(\ell), t\theta(\ell+1))$. By the definition of $\theta(\ell)$, we have $\theta(\ell+1) < p^{\ell+1}$. Thus $r < t\theta(\ell+1) < tp^{\ell+1}$. It follows that $\ell > \log_p \frac{r}{t} - 1$. By Theorem 4.3, we have

$$\begin{aligned}\mathsf{D}(G) &\geq \mathsf{D}^*(G) + t((p-1)\ell - m) \\ &> \mathsf{D}^*(G) + t\left((p-1)\left(\log_p \frac{r}{t} - 1\right) - m\right) \\ &= \mathsf{D}^*(G) + \frac{t(p-1)}{\log p} \log r - t(p-1)(\log_p t + 1) - tm.\end{aligned}$$

□

Remark 4.5. *Let $G = C_n^r \oplus C_{kn}$ be a non-p-group with $n, k \in \mathbb{N}_{\geq 2}$. In Corollary 4.4, let $t = 1$, we have*

$$\mathsf{D}(G) > \mathsf{D}^*(G) + (p-1)\log_p r - m - p + 1. \tag{9}$$

So $\mathsf{D}(G) - \mathsf{D}^*(G)$ grows at least logarithmically with respect to r. And this inequality does not depend on the size of k_1. That is to say, it can be $\mathsf{D}(G) - \mathsf{D}^*(G) > 0$ for arbitrarily large exponent of G.

We have $\mathsf{D}(G) - \mathsf{D}^*(G) > t\left((p-1)\left(\log_p \frac{r}{t} - 1\right) - m\right)$ by Corollary 4.4. Fix p and m. Let r be larger than some constant, by (9), then there always exists $t \in [1, k_1 - 1]$ such that $\mathsf{D}(G) - \mathsf{D}^*(G) > 0$. Let $t = c_1 r$, where $c_1 \in (0,1)$ is a real number such that $(p-1)\left(\log_p \frac{r}{t} - 1\right) - m > 0$. Then for sufficiently large $k_1 = k_1(r)$ such that $t \in [1, k_1]$, by Corollary 4.4, we always have $\mathsf{D}(G) - \mathsf{D}^*(G) > c_2 r$, where $c_2 > 0$ is a constant determined by p, m and c_1. Note that c_1 is bounded by p and m. See (12) for more information about $\frac{\mathsf{D}(G) - \mathsf{D}^*(G)}{r}$.

On the other hand, fix n and k, for sufficiently large r, we can let $t = k_1 - 1$ and p be as large as possible to get larger $\mathsf{D}(G) - \mathsf{D}^*(G)$ in (8).

Next, we give a general lower bound to abelian non-p-groups and express the lower bound of $\mathsf{D}(G) - \mathsf{D}^*(G)$ by the rank and the exponent of G. In Theorem 4.6, we define $\log(0) = -\infty$ for the case of $|G| = m^r$.

Theorem 4.6. *Let G be a finite abelian non-p-group of rank $r \in \mathbb{N}_+$ and exponent $m \in \mathbb{N}_{\geq 2}$. Then*

$$\mathsf{D}(G) \geq \mathsf{D}^*(G) + \max\{\log_2 \log \frac{m^r}{|G|} - 2\log_2 \log \frac{m}{2} - m + \log_2 \log 2 + 1,\ 0\}.$$

Proof. $\mathsf{D}(G) \geq \mathsf{D}^*(G)$ is trivial.

Note that any abelian non-p-group G's exponent $m \geq 6$. So $\log\log \frac{m}{2} > 0$. If $|G| = m^r$, since we define that $\log(0) = -\infty$, the inequality in this theorem holds.

Suppose that $|G| \neq m^r$ and

$$G = C_{n_1}^{x_1} \oplus \cdots \oplus C_{n_t}^{x_t} \oplus C_m^x$$

with $n_1 | \ldots | n_t | m$ and $1 < n_1 < \cdots < n_t < m$. Let $x_a = \max\{x_i, i \in [1,t]\}$. By Lemma 2.3, (9) and $\frac{p-1}{\log p} \geq \frac{1}{\log 2}$, we have

$$\mathsf{D}(G) > \mathsf{D}^*(G) + \log_2 x_a - m + 1. \tag{10}$$

Since $m \geq 2n_t \geq 2^2 n_{t-1} \geq \cdots \geq 2^t n_1$, we have $t \leq \log_2 \frac{m}{n_1}$. Together with $x_a t \geq x_1 + \cdots + x_t = r - x$. We derive that

$$x_a \geq \frac{r-x}{\log_2 \frac{m}{n_1}}. \tag{11}$$

By

$$\frac{m^r}{|G|} = \frac{m^r}{n_1^{x_1} n_2^{x_2} \dots n_t^{x_t} m^x} \leq \left(\frac{m}{n_1}\right)^{r-x},$$

we have $r - x \geq \log_{\frac{m}{n_1}} \frac{m^r}{|G|}$. Together with (11), we have

$$x_a \geq \frac{\log_{\frac{m}{n_1}} \frac{m^r}{|G|}}{\log_2 \frac{m}{n_1}} = \frac{\log \frac{m^r}{|G|} \log 2}{\log^2 \frac{m}{n_1}}.$$

Then by (10), it follows that

$$\begin{aligned} \mathsf{D}(G) &> \mathsf{D}^*(G) + \log_2 \frac{\log \frac{m^r}{|G|} \log 2}{\log^2 \frac{m}{n_1}} - m + 1 \\ &\geq \mathsf{D}^*(G) + \log_2 \log \frac{m^r}{|G|} - 2\log_2 \log \frac{m}{2} - m + \log_2 \log 2 + 1. \end{aligned}$$

Thus the theorem is proved. □

So far, all the known groups G with $\mathsf{D}(G) - \mathsf{D}^*(G) > 0$ are non-p-groups satisfying $|G| < \exp(G)^{r(G)}$. We would like to generalize this to a corollary as follows.

Corollary 4.7. *Given a non-prime power $m > 0$. Let G be abelian groups with exponent m and rank r, then for each $N > 0$ there exists an $\varepsilon = \varepsilon(N; m) > 0$ such that if $\frac{|G|}{m^r} < \varepsilon$, then $\mathsf{D}(G) - \mathsf{D}^*(G) > N$.*

Proof. This follows directly from Theorem 4.6. □

Remark 4.8. *Let $G = C_n^r \oplus C_{kn}$ be a non-p-group with $n, k \in \mathbb{N}_{\geq 2}$, we can consider the small rank r such that $\mathsf{D}(G) > \mathsf{D}^*(G)$. Theorem 4.3 shows that if $(p-1)\ell - m > 0$, then $\mathsf{D}(G) > \mathsf{D}^*(G)$. Thus, let $\ell = \lfloor \frac{m}{p-1} \rfloor + 1$. And $r = \theta(\lfloor \frac{m}{p-1} \rfloor + 1)$ is a small r such that $\mathsf{D}(G) > \mathsf{D}^*(G)$.*

The groups G of *small rank* with $\mathsf{D}(G) > \mathsf{D}^*(G)$ were viewed as "*the interesting groups*" on page 148 in [12]. We give following corollary about the small rank.

Corollary 4.9. *1) Let $G = C_p^r \oplus C_{kp}$ with p odd prime and $\gcd(p, k) = 1$. If $r \geq 2p$, then $\mathsf{D}(G) - \mathsf{D}^*(G) \geq p - 2 > 0$. Thus*

$$\sup_{\text{all finite abelian group } G} \frac{\mathsf{D}(G) - \mathsf{D}^*(G)}{r} \geq \frac{1}{2}. \tag{12}$$

2) Let $G = C_2^r \oplus C_{2^t k}$ with $k > 2$ odd and integer $t \geq 1$. If $r \geq 2^{2^t+1} - 2^t - 2$, then $\mathsf{D}(G) \geq \mathsf{D}^*(G) + 1$.

Proof. 1) Let $\ell = 2$, then $\theta(\ell) = 2p$. By Theorem 4.3, if $r \geq 1 \cdot \theta(\ell) = 2p$, then $\mathsf{D}(G) - \mathsf{D}^*(G) \geq (p-1)\ell - p = p - 2 > 0$.

2) Let $\ell = 2^t + 1$ and $p = 2$, then $\theta(\ell) = 2^{2^t+1} - 2 - 2^t$. By Theorem 4.3, if $r \geq \theta(\ell)$, then $\mathsf{D}(G) \geq \mathsf{D}^*(G) + \ell - 2^t = \mathsf{D}^*(G) + 1$.

□

In particular, let $G = C_2^r \oplus C_{2k}$ with $k \geq 3$ odd. If $r \geq 4$, then $\mathsf{D}(G) - \mathsf{D}^*(G) \geq 1$. Note that for abelian group $G = C_2^4 \oplus C_{2k}$ with odd $k \geq 70$, it is proved that $\mathsf{D}(G) = \mathsf{D}^*(G) + 1$ (see [20]). In addition, it is interesting to determine $\sup \frac{\mathsf{D}(G) - \mathsf{D}^*(G)}{r}$, where G runs over all finite abelian groups.

5. Concluding remarks

Open problem. *By Lemma 2.3, a natural question occurs. What are the groups G, with the invariant factor decomposition*

$$G = C_{n_1} \oplus C_{n_2} \oplus \cdots \oplus C_{n_r} \text{ with } 1 < n_1 | n_2 \ldots | n_r,$$

such that there do not exist groups

$$H_x = \oplus_{i \in I_x} C_{n_i}, \text{ with } \varnothing \neq I_x \subsetneq [1, r] \text{ and } I_x \cap I_y = \varnothing \text{ for any } x, y \in [1, z],$$

satisfying that $\mathsf{D}(G) - \mathsf{D}^*(G) = \sum_{x=1}^{z} (\mathsf{D}(H_x) - \mathsf{D}^*(H_x))$.

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References

[1] G. Bhowmik and J.-C. Schlage-Puchta, *Davenports constant for groups of the form $Z_3 \oplus Z_3 \oplus Z_{3d}$*, Additive Combinatorics (A. Granville, M.B. Nathanson, and J. Solymosi, eds.), CRM Proceedings and Lecture Notes, vol. 43, American Mathematical Society, 2007, pp. 307–326.

[2] G. Bhowmik and J.-C. Schlage-Puchta, *Davenport's constant for groups with large exponent*, Theory and applications of finite fields, Contemporary Mathematics, vol. 579, AMS, 2013, pp. 21–31.

[3] P. van Emde Boas, *A combinatorial problem on finite abelian groups II*, Reports ZW-1969-007, Mathematical Centre, Amsterdam, 1969.

[4] P. Erdős and E. Szemerédi, *On a problem of Graham*, Publ. Math. Debrecen. **23** (1976), 123–127.

[5] W. Gao, Y. Hamidoune and G. Wang, *Distinct lengths modular zero-sum subsequences: a proof of Graham's conjecture*, J. Number Theory **106** (2010) 1425–1431.

[6] W. Gao and A. Geroldinger, *On long minimal zero sequences in finite abelian groups*, Period. Math. Hung. **38** (1999), 179–211.

[7] W. Gao and A. Geroldinger, *Zero-sum problems in finite abelian groups : a survey*, Expo. Math. **24** (2006), 337–369.

[8] W. Gao, P. Zhao and J. Zhuang, *Zero-sum subsequences of distinct lengths*, Int. J. Number Theory **11(07)** (2015), 2141–2150.

[9] A. Geroldinger, *Additive group theory and non-unique factorizations*, Combinatorial Number Theory and Additive Group Theory, Advanced Courses in Mathematics CRM Barcelona, Birkhâuser, 2009, pp. 1–86.

[10] A. Geroldinger and F. Halter-Koch, *Non-Unique Factorizations. Algebraic, Combinatorial and Analytic Theory*, Pure and Applied Mathematics 278, Chapman & Hall/CRC 2006.

[11] A. Geroldinger, M. Liebmann and A. Philipp, *On the Davenport constant and on the structure of extremal zero-sum free sequences*, Periodica Mathematica Hungarica **64(2)** (2012), 213–225.

[12] A. Geroldinger and R. Schneider, *On Davenports constant*, J. Comb. Theory Ser. A **61** (1992), 147–152.

[13] B. Girard, *An asymptotically tight bound for the Davenport constant*, J. Ec. polytech. Math. **5** (2018), 605–611.

[14] B. Girard, *On the existence of zero-sum subsequences of distinct lengths*, Rocky Mountain J. Math. **42** (2012), 583–596.

[15] B. Girard and W.A. Schmid, *Direct zero-sum problems for certain groups of rank three*, J. Number Theory, https://doi.org/10.1016/j.jnt.2018.08.016.

[16] B. Girard and W.A. Schmid, *Inverse zero-sum problems for certain groups of rank three*, https://arxiv.org/abs/1809.03178.

[17] D.J. Grynkiewicz, *Note on a conjecture of Graham*, European J. Combin. **32** (2011), 1334–1344.

[18] D.J. Grynkiewicz, *Structural Additive Theory*, Developments in Mathematics 30, Springer, Cham, 2013.

[19] M. Mazur, *A note on the growth of Davenport's constant*, Manuscripta Mathematica **74(1)** (1992), 229–235.

[20] S. Savchev and F. Chen, *Long minimal zero-sum sequences in the group $C_2^{r-1} \oplus C_{2k}$*, Integers (2012), A51.

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