ON THE INDEX-R-FREE SEQUENCES OVER FINITE CYCLIC GROUPS

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ABSTRACT. Let C_n be a finite cyclic group of order $n \geq 2$. Every sequence S over C_n can be written in the form $S = (n_1 g), \ldots, (n_l g)$ where $g \in C_n$ and $n_1, \ldots, n_l \in [1, \operatorname{ord}(g)]$, and the index $\operatorname{ind}(S)$ of S is defined as the minimum of $(n_1 + \ldots + n_l)/\operatorname{ord}(g)$ over all $g \in C_n$ with $\operatorname{ord}(g) = n$. Let $d > 1$ and $r \geq 1$ be any fixed integers. We prove that, for every sufficiently large integer n divisible by d, there exists a sequence S over C_n of length $|S| \geq n + n/d + O(\sqrt{n})$ having no subsequence T of index $\operatorname{ind}(T) \in [1, r]$, which has substantially improved the previous results in this direction.

1. Introduction and Main Results

Throughout this paper, let C_n be an additively written finite cyclic group of order $|C_n| = n$, where $n \in \mathbb{Z}$ with $n > 1$. By a *sequence* S of *length* $|S| = \ell$ over C_n we mean an unordered sequence with ℓ terms from C_n and the repetition of terms is allowed. We call S a *zero-sum sequence* if the sum of S is zero. We let $\mathbb{Z}$ denote the integers, and $\mathbb{R}$ the real numbers. Given real numbers $a, b \in \mathbb{R}$, we use $[a, b] := \{u : u \in \mathbb{Z},\ a \leq u \leq b\}$ to denote all integers between a and b. Recall that the index of a sequence S is defined as follows.

Definition 1.1. *For a sequence*

$$S = (n_1 g) \cdot \ldots \cdot (n_l g) \quad \textit{over} \quad C_n,$$

where $n_1, \ldots, n_l \in [1, n]$ *and* $g \in C_n$ *with* $\operatorname{ord}(g) = |C_n|$, *we set*

$$\|S\|_g = \frac{n_1 + \ldots + n_l}{n},$$

and the index of S *is defined by*

$$\operatorname{ind}(S) = \min\{\|S\|_g \mid g \in C_n \textit{ with } \operatorname{ord}(g) = |C_n|\}.$$

The index of a sequence is a crucial invariant in the investigation of zero-sum sequences over cyclic groups. It was first addressed by Lemke and Kleitman ([9]), used as a key tool by Geroldinger ([7, page 736]), and then investigated by Gao [3] in a systematical way. And it has found a lot of attention in recent years (see [1, 2, 4, 6, 8, 10, 11, 13, 15, 16]). If S is a minimal zero-sum sequence, then $|S| \leq 3$, as well as $|S| \geq \lfloor \frac{n}{2} \rfloor + 2$, implies that $\operatorname{ind}(S) = 1$ (see [1], [12], [14]).

An important open problem (at the end of [5]) is to determine the maximum length of sequences over C_n without index 1 subsequences. Clearly, S is a zero-sum sequence if and only if $\operatorname{ind}(S)$ is an integer by definition 1.1. Hence we introduce the definitions of $\mathsf{t}_r(n)$ and *index-r-free* sequences.

Definition 1.2. *Let* r *be a positive integer, denote by* $\mathsf{t}_r(n)$ *the smallest integer* ℓ *such that every sequence* S *over* C_n *of length* $|S| \geq \ell$ *has a zero-sum subsequence* T *with* $\operatorname{ind}(T) \in [1, r]$.

Definition 1.3. *For any integer $r \geq 1$, a sequence S over C_n is called index-r-free, if S has no zero-sum subsequence T with* $\operatorname{ind}(T) \in [1, r]$.

In 1989, Lemke and Kleitman ([9, page 344]) conjectured that if S is a sequence over C_n of length $|S| = n$, then there exists a subsequence T of S such that $\operatorname{ind}(T) = 1$. That is to say, $\mathsf{t}_1(n) = n$. In 2011, Gao, Li, Peng, Plyley and Wang ([5]) gave a counterexample and proved that $\mathsf{t}_1(n) \geq n + \lfloor \frac{n}{4} \rfloor - 4$ for $n = 4k + 2 \geq 22$. In 2015, Zeng, Yuan and Li ([16]) promoted the former counterexample to general counterexamples, and by their results we could derive that $\mathsf{t}_1(n) \geq n + \lfloor \frac{n}{d^2} \rfloor - (d^3 - d^2 + d - 1)$ for $n > d^2(d^3 - d^2 + d + 1)$, where $d \in \mathbb{Z}$ with $d > 1$.

In this paper we give longer general structures (theorem 1.4) to the conjecture of Lemke and Kleitman, and prove that $\mathsf{t}_1(n) \geq n + \frac{n}{d} + O(\sqrt{n})$ for every sufficiently large integer n divisible by d, where $d \in \mathbb{Z}$ with $d > 1$ (theorem 1.5). It is a greater lower bound of $\mathsf{t}_1(n)$ than before, and we conjecture that it is the best possible bound when n is big enough. Furthermore, we promote the index 1 free sequences to index r free sequences, and show that $\mathsf{t}_{\mathbf{r}}(n) \geq n + \frac{n}{d} + O(\sqrt{n})$ for every sufficiently large integer n divisible by d, where constant $r \in \mathbb{Z}$ with $r \geq 2$. Here are our main results.

Theorem 1.4. *Let d, n be any integers with $1 < d | n$ and $n > d^2$, and $g \in C_n$ with* $\operatorname{ord}(g) = n$. *For every integer $r \in \left[1, \frac{n}{d^2}\right)$ and $k \in \left[0, \log_d^{\frac{n}{r}} - 2\right)$,*

$$S = \prod_{(i,j) \in A} \Big(\left(im + d^j\right) g \Big)^{\left\lfloor \frac{m}{d^j} \right\rfloor - (dr-1)d^{k-j} - 1} \tag{1}$$

is an index-r-free sequence, where $m = \frac{n}{d}$ and $A = [1, d-1] \times [0, k] \bigcup \{(0, 0)\}$.

Theorem 1.5. *Given any fixed integers $d > 1$ and $r \geq 1$, for every sufficiently large integer n with $d | n$, there exists an index-r-free sequence S over C_n such that $|S| \geq n + \frac{n}{d} + O(\sqrt{n})$.*

In the following sections we provide the preliminaries and the proofs of Theorem 1.4 and Theorem 1.5. We end the paper with a further conjecture and an open problem.

2. Notations and Preliminaries

We let n and d be any integers with $1 < d | n$ and $n > d^2$, and let $g \in C_n$ with $\operatorname{ord}(g) = n$. For every integer $r \in \left[1, \frac{n}{d^2}\right)$ and $k \in \left[0, \log_d^{\frac{n}{r}} - 2\right)$, let a sequence

$$S = \prod_{(i,j) \in A} \Big(\left(im + d^j\right) g \Big)^{\left\lfloor \frac{m}{d^j} \right\rfloor - (dr-1)d^{k-j} - 1},$$

where $m = \frac{n}{d}$ and $A = [1, d-1] \times [0, k] \bigcup \{(0, 0)\}$.

Let T be a subsequence of S and $t_{ij} \in \mathbb{Z}$ be the multiplicity of $(im + d^j)g$ in T, where $(i, j) \in A$. If $(im + d^j)g \notin T$, we set $t_{ij} = 0$. That is,

$$T = \prod_{(i,j) \in A} \left((im + d^j) g \right)^{t_{ij}} \subset S,$$

where

$$0 \le t_{ij} \le \left\lfloor \frac{m}{d^j} \right\rfloor - (dr-1)d^{k-j} - 1. \tag{2}$$

We set $\mathrm{ind}(T) = \parallel T \parallel_{g_1}$, where $g_1 \in C_n$ with $\langle g_1 \rangle = C_n$. And we set $g = hg_1$, where $h \in [1, n-1]$ with $\gcd(h,n) = 1$. Then

$$T = \prod_{(i,j)\in A} \left((im + d^j)hg_1 \right)^{t_{ij}},$$

and

$$n \parallel T \parallel_{g_1} = \sum_{(i,j)\in A} t_{ij} \left| (im + d^j)h \right|_n, \tag{3}$$

where $\left|w\right|_n$ denotes the least positive residue of $w \in \mathbb{Z}$ modulo $n > 0$. We fix the notation concerning sequences over C_n. And let

$$B = \left\{ (i,j) \in A \,\middle|\, 0 < \left|(im + d^j)h\right|_n < m \right\},$$

and

$$C = \left\{ (i,j) \in A \,\middle|\, m < \left|(im + d^j)h\right|_n < n \right\}.$$

By next lemma we split A into two parts.

Lemma 2.1. *$B \cup C = A$.*

Proof. For every $(i,j) \in A$, combining $A = [1, d-1] \times [0,k] \bigcup \{(0,0)\}$, $r \in \left[1, \frac{n}{d^2}\right)$ with $k \in \left[0, \log_d^{\frac{n}{r}} - 2\right)$, we derive $0 < d^j < m$. Then by $\gcd(h,n) = 1$ and $dm = n$, we have $0 < \left|(im + d^j)h\right|_n < n$ and $\left|(im + d^j)h\right|_n \ne m$ for every $(i,j) \in A$. Then by the definitions of B and C, we have $B \cup C = A$. □

Lemma 2.2. *For any integer $j \in [0,k]$, we have*

$$\left\{ \left|(im + d^j)h\right|_n \,\middle|\, i \in [0, d-1] \right\} = \left\{ im + \left|hd^j\right|_m \,\middle|\, i \in [0, d-1] \right\},$$

and there exists only one element $i_0 \in [0, d-1]$ such that $0 < \left|(i_0 m + d^j)h\right|_n < m$.

Proof. By

$$\left| \left|(im + d^j)h\right|_n \right|_m = \left|hd^j\right|_m, \text{ where } i \in [0, d-1],$$

we have

$$\left\{ \left|(im + d^j)h\right|_n \,\middle|\, i \in [0, d-1] \right\} \subset \left\{ im + \left|hd^j\right|_m \,\middle|\, i \in \mathbb{Z} \right\}.$$

For any $j \in [0,k]$, by the relevant definitions we have $0 < d^j < m$, then $0 < \left|(im + d^j)h\right|_n < n$. So we have

$$\left\{ \left|(im + d^j)h\right|_n \,\middle|\, i \in [0, d-1] \right\} \subset \left\{ im + \left|hd^j\right|_m \,\middle|\, i \in [0, d-1] \right\}.$$

By $\gcd(h,n) = 1$, we derive that $\left\{ \left|(im + d^j)h\right|_n \,\middle|\, i \in [0, d-1] \right\}$ have d distinct elements. Since these two sets both have d elements, we have

$$\left\{ \left|(im + d^j)h\right|_n \,\middle|\, i \in [0, d-1] \right\} = \left\{ im + \left|hd^j\right|_m \,\middle|\, i \in [0, d-1] \right\},$$

and there exists only one element $i_0 \in [0, d-1]$ such that

$$0 < \left|(i_0 m + d^j)h\right|_n < m.$$

□

By lemma 2.1, we rewrite Eq. (3) as

$$n \parallel T \parallel_{g_1} = \left(\sum_{(i,j)\in B} + \sum_{(i,j)\in C} \right) t_{ij} \left| \left(im + d^j\right) h \right|_n . \tag{4}$$

We consider the d elements of A, $(i,0)$, where $i \in [0, d-1]$. By lemma 2.2, we have

$$\left\{ \left|(im + d^0)h\right|_n \,\middle|\, i \in [0, d-1] \right\} = \left\{ im + \left|hd^0\right|_m \,\middle|\, i \in [0, d-1] \right\}.$$

Then for some $i_0 \in [0, d-1]$, one has $\left|(i_0 m + d^0)h\right|_n = |h|_m$, so $(i_0, 0) \in B$. For some $i_1 \in [0, d-1]$, one has $\left|(i_1 m + d^0)h\right|_n = m + |h|_m$, so $(i_1, 0) \in C$. Then we derive that $B, C \neq \emptyset$. Here we set $|B| = x$ and sort the elements in B as

$$B = \{ (\mu_1, \tau_1), (\mu_2, \tau_2), \cdots, (\mu_x, \tau_x) \},$$

where μ_*, τ_* and x are integers with $\mu_* \in [0, d-1]$, $0 = \tau_1 \leq \tau_2 \leq \cdots \leq \tau_x \leq k$ and $x \geq 1$.

By lemma 2.2, we derive that for any integer τ_*, there exists at most one element $\mu_* \in [0, d-1]$ such that $0 < \left|(\mu_* m + d^{\tau_*})h\right|_n < m$. By the enumeration of the elements of B, we know that actually $0 = \tau_1 < \tau_2 < \cdots < \tau_x \leq k$.

Next we will prove another quality of the sorted elements in B when $x \geq 2$.

Lemma 2.3. *When $|B| = x \geq 2$, for every integer $a \in [1, x-1]$, we have*

$$m < \left|(\mu_a m + d^{\tau_a})h\right|_n d^{\tau_{a+1} - \tau_a} < n.$$

Proof. **Case 1.** $\tau_{a+1} - \tau_a = 1$.

By the definition of B we have $0 < \left|(\mu_a m + d^{\tau_a})h\right|_n < m$, thus $0 < \left|(\mu_a m + d^{\tau_a})h\right|_n d < n$. It is clear that $\left|(\mu_a m + d^{\tau_a})h\right|_n d \neq m$. Assuming that $0 < \left|(\mu_a m + d^{\tau_a})h\right|_n d < m$, by the definition of B we also have $0 < \left|(\mu_{a+1} m + d^{\tau_{a+1}})h\right|_n < m$. Thus

$$\left|(\mu_a m + d^{\tau_a})h\right|_n d - \left|(\mu_{a+1} m + d^{\tau_{a+1}})h\right|_n \in (-m, m). \tag{5}$$

But we have

$$\left| \left|(\mu_a m + d^{\tau_a})h\right|_n d - \left|(\mu_{a+1} m + d^{\tau_{a+1}})h\right|_n \right|_n = \left| -\mu_{a+1} hm \right|_n = \left| -\mu_{a+1} h \right|_d m .$$

Since $\mu_{a+1} \in [1, d-1]$ and $\gcd(h, n) = 1$, we have $\left| -\mu_{a+1} h \right|_d \neq d$. Hence

$$\left|(\mu_a m + d^{\tau_a})h\right|_n d - \left|(\mu_{a+1} m + d^{\tau_{a+1}})h\right|_n = ym \text{ with integer } y \neq 0,$$

a contradiction to Eq. (5). So that $m < \left|(\mu_a m + d^{\tau_a})h\right|_n d < n$.

Case 2. $\tau_{a+1} - \tau_a \geq 2$.

First, for any integers $v \in [\tau_a + 1, \tau_{a+1} - 1]$ and $i \in [1, d-1]$, we have $(i, v) \in A$ by the definition of A. By definition of B, $(i, v) \notin B$. By lemma 2.1, we have $(i, v) \in C$. Then by the definition of C, we have

$$m < \left|(im + d^v)h\right|_n < n, \tag{6}$$

where $v \in [\tau_a + 1, \tau_{a+1} - 1]$ and $i \in [1, d-1]$.

Second, for every $z \in [0\,,\, \tau_{a+1} - \tau_a - 2]$, we will prove that, if $0 < \left|(\mu_a m + d^{\tau_a})h\right|_n d^z < m$, then $0 < \left|(\mu_a m + d^{\tau_a})h\right|_n d^{z+1} < m$.

For every $z \in [0\,,\, \tau_{a+1} - \tau_a - 2]$, we let $v = \tau_a + z + 1$, and suppose that

$$0 < \left|(\mu_a m + d^{\tau_a})h\right|_n d^z < m.$$

Then we have

$$0 < \left|(\mu_a m + d^{\tau_a})h\right|_n d^{z+1} < n.$$

Therefore,

$$\left|(\mu_a m + d^{\tau_a})h\right|_n d^{z+1} = \left|(\mu_a m + d^{\tau_a})hd^{z+1}\right|_n = \left|d^{\tau_a + z+1}h\right|_n = \left|hd^v\right|_n . \tag{7}$$

By lemma 2.2, we have

$$\left\{\left|(im + d^v)h\right|_n \,\middle|\, i \in [\mathbf{0}, d-1]\right\} = \left\{im + \left|hd^v\right|_m \,\middle|\, i \in [\mathbf{0}, d-1]\right\}. \tag{8}$$

Note that $v = \tau_a + z + 1 \in [\tau_a + 1,\, \tau_{a+1} - 1]$. By Eq. (6), we have

$$\left\{\left|(im + d^v)h\right|_n \,\middle|\, i \in [\mathbf{1}, d-1]\right\} \subset \left\{im + \left|hd^v\right|_m \,\middle|\, i \in [\mathbf{1}, d-1]\right\}.$$

Since these two sets both have $d-1$ elements, we have

$$\left\{\left|(im + d^v)h\right|_n \,\middle|\, i \in [\mathbf{1}, d-1]\right\} = \left\{im + \left|hd^v\right|_m \,\middle|\, i \in [\mathbf{1}, d-1]\right\}. \tag{9}$$

Then combining Eq. (8) with Eq. (9), we have

$$\left\{\left|(im + d^v)h\right|_n \,\middle|\, i = 0\right\} = \left\{im + \left|hd^v\right|_m \,\middle|\, i = 0\right\}.$$

That is, $\left|hd^v\right|_n = \left|hd^v\right|_m$. Then by $0 < \left|hd^v\right|_m < m$ and Eq. (7), we have

$$0 < \left|(\mu_a m + d^{\tau_a})h\right|_n d^{z+1} < m.$$

Last, thus we proceed by induction on $z \in [0\,,\, \tau_{a+1} - \tau_a - 2]$. Since $0 < \left|(\mu_a m + d^{\tau_a})h\right|_n d^z < m$ is true for $z = 0$ by the definition of B, we let $z = \tau_{a+1} - \tau_a - 2$ and derive that

$$0 < \left|(\mu_a m + d^{\tau_a})h\right|_n d^{\tau_{a+1} - \tau_a - 1} < m$$

is true. Thus $0 < \left|(\mu_a m + d^{\tau_a})h\right|_n d^{\tau_{a+1} - \tau_a} < n$. It is clear that $\left|(\mu_a m + d^{\tau_a})h\right|_n d^{\tau_{a+1} - \tau_a} \neq m$. Assuming that $0 < \left|(\mu_a m + d^{\tau_a})h\right|_n d^{\tau_{a+1} - \tau_a} < m$, by the definition of B we also have $0 < \left|(\mu_{a+1} m + d^{\tau_{a+1}})h\right|_n < m$. Thus

$$\left|(\mu_a m + d^{\tau_a})h\right|_n d^{\tau_{a+1} - \tau_a} - \left|(\mu_{a+1} m + d^{\tau_{a+1}})h\right|_n \in (-m, m). \tag{10}$$

But we have

$$\left|\,\left|(\mu_a m + d^{\tau_a})h\right|_n d^{\tau_{a+1} - \tau_a} - \left|(\mu_{a+1} m + d^{\tau_{a+1}})h\right|_n\right|_n = \left| -\mu_{a+1} h\right|_d m\,.$$

It is a contradiction to Eq. (10). So that $m < \left|(\mu_a m + d^{\tau_a})h\right|_n d^{\tau_{a+1} - \tau_a} < n$. □

3. Proof of Theorem 1.4 and Theorem 1.5

Proof of Theorem 1.4. Suppose to the contrary that there exists a subsequence $T \subset S$ with $T \neq \emptyset$ and $\mathrm{ind}(T) \in [1, r]$. We use the same relevant notions defined in last section. Without loss of generality, we assume that $|B| = x \geq 2$, because the following proof also holds true by some minor modifications (for example, we view all the $\sum_{l=1}^{x-1} f(l)$ as 0 when $x = 1$). We could rewrite Eq. (4) as

$$\begin{aligned} n \parallel T \parallel_{g_1} = &\sum_{l=1}^{x-1} t_{\mu_l\, \tau_l} \left|(\mu_l\, m + d^{\tau_l})h\right|_n + t_{\mu_x\, \tau_x} \left|(\mu_x\, m + d^{\tau_x})h\right|_n \\ &+ \sum_{(i,j)\in C} t_{ij} \left|(im + d^j)h\right|_n . \end{aligned} \tag{11}$$

For $l \in [1, x-1]$, we set

$$t_{\mu_l\, \tau_l} = s_l\, d^{\tau_{l+1} - \tau_l} + t'_{\mu_l\, \tau_l}\, , \tag{12}$$

where $s_l \geq 0$ and $t'_{\mu_l\, \tau_l} \in [0\, ,\, d^{\tau_{l+1} - \tau_l} - 1]$. Then we use three steps to complete the proof.

First, we will prove that $\sum_{l=1}^{x-1} s_l + \sum_{(i,j)\in C} t_{ij} \leq dr - 1$. By Eqs. (11) and (12), we have

$$\begin{aligned} n \parallel T \parallel_{g_1} = &\sum_{l=1}^{x-1} (s_l\, d^{\tau_{l+1} - \tau_l} + t'_{\mu_l\, \tau_l}) \left|(\mu_l\, m + d^{\tau_l})h\right|_n \\ &+ t_{\mu_x\, \tau_x} \left|(\mu_x\, m + d^{\tau_x})h\right|_n + \sum_{(i,j)\in C} t_{ij} \left|(im + d^j)h\right|_n \\ = &\sum_{l=1}^{x-1} t'_{\mu_l\, \tau_l} \left|(\mu_l\, m + d^{\tau_l})h\right|_n + t_{\mu_x\, \tau_x} \left|(\mu_x\, m + d^{\tau_x})h\right|_n \\ &+ \Bigg(\sum_{l=1}^{x-1} s_l\, d^{\tau_{l+1} - \tau_l} \left|(\mu_l\, m + d^{\tau_l})h\right|_n \\ &+ \sum_{(i,j)\in C} t_{ij} \left|(im + d^j)h\right|_n \Bigg). \end{aligned} \tag{13}$$

Hence we have

$$\begin{aligned} n \parallel T \parallel_{g_1} \geq &\sum_{l=1}^{x-1} s_l\, d^{\tau_{l+1} - \tau_l} \left|(\mu_l m + d^{\tau_l})h\right|_n + \sum_{(i,j)\in C} t_{ij} \left|(im + d^j)h\right|_n \\ > &\sum_{l=1}^{x-1} s_l\, m + \sum_{(i,j)\in C} t_{ij}\, m = \left(\sum_{l=1}^{x-1} s_l + \sum_{(i,j)\in C} t_{ij}\right) m\, . \end{aligned}$$

We suppose that $\sum_{l=1}^{x-1} s_l + \sum_{(i,j)\in C} t_{ij} > dr$, and derive

$$n \parallel T \parallel_{g_1} > \left(\sum_{l=1}^{x-1} s_l + \sum_{(i,j)\in C} t_{ij}\right) m > rn.$$

Thus $\mathrm{ind}(T) = \parallel T \parallel_{g_1} > r$, a contradiction to $\mathrm{ind}(T) \in [1, r]$. So we have

$$\sum_{l=1}^{x-1} s_l + \sum_{(i,j)\in C} t_{ij} \leq dr - 1. \tag{14}$$

Next, we will prove that $\left| n \parallel T \parallel_{g_1} \right|_{\mathbf{m}} \neq m$. By Eq. (13), we have

$$\begin{aligned}
&\left| n \parallel T \parallel_{g_1} \right|_{\mathbf{m}} \\
=&\left| \sum_{l=1}^{x-1} t'_{\mu_l\,\tau_l}\, d^{\tau_l} h + t_{\mu_x\,\tau_x}\, d^{\tau_x} h + \sum_{l=1}^{x-1} s_l\, d^{\tau_{l+1}-\tau_l}\, d^{\tau_l} h + \sum_{(i,j)\in C} t_{ij}\, d^j h \right|_{\mathbf{m}} \\
=&\left| h\Big(\sum_{l=1}^{x-1} t'_{\mu_l\,\tau_l}\, d^{\tau_l} + t_{\mu_x\,\tau_x}\, d^{\tau_x} + \sum_{l=1}^{x-1} s_l\, d^{\tau_{l+1}} + \sum_{(i,j)\in C} t_{ij}\, d^j \Big) \right|_{\mathbf{m}} \\
=&\left| h(**) \right|_{\mathbf{m}}, \qquad (15)
\end{aligned}$$

where

$$\begin{aligned}
(**) =& \sum_{l=1}^{x-1} t'_{\mu_l\,\tau_l}\, d^{\tau_l} + t_{\mu_x\tau_x}\, d^{\tau_x} + \sum_{l=1}^{x-1} s_l\, d^{\tau_{l+1}} + \sum_{(i,j)\in C} t_{ij}\, d^j \\
\leq& \sum_{l=1}^{x-1} (d^{\tau_{l+1}-\tau_l} - 1) d^{\tau_l} + t_{\mu_x\tau_x}\, d^{\tau_x} + \sum_{l=1}^{x-1} s_l\, d^{\mathbf{k}} + \sum_{(i,j)\in C} t_{ij}\, d^{\mathbf{k}} \\
=& - d^{\tau_1} + d^{\tau_x} + t_{\mu_x\tau_x}\, d^{\tau_x} + \left(\sum_{l=1}^{x-1} s_l + \sum_{(i,j)\in C} t_{ij} \right) d^k \\
\leq& - d^{\tau_1} + d^{\tau_x} + \left(\left\lfloor \frac{m}{d^{\tau_x}} \right\rfloor - (dr-1) d^{k-\tau_x} - 1 \right) d^{\tau_x} + (dr-1) d^k \qquad (16) \\
\leq& - d^{\tau_1} + d^{\tau_x} + m - (dr-1) d^k - d^{\tau_x} + (dr-1) d^k \\
\leq& m - 1\,. \qquad (17)
\end{aligned}$$

It is clear that $(**) > 0$ by $T \neq \emptyset$. So we have $\left| n \parallel T \parallel_{g_1} \right|_{\mathbf{m}} = \left| h(**) \right|_{\mathbf{m}} \neq m$ by Eqs. (15) and (17).

Last, since $\left| n \parallel T \parallel_{g_1} \right|_{\mathbf{m}} \neq m$ and $m|n$, we have $\left| n \parallel T \parallel_{g_1} \right|_{\mathbf{n}} \neq n$. Hence $\mathrm{ind}(T) = \parallel T \parallel_{g_1}$ is not an integer and T is not a zero-sum subsequence of S. It is a contradiction to $\mathrm{ind}(T) \in [1, r]$. Thus S is an index-r-free sequence.

□

Proof of Theorem 1.5. Given any fixed integers $d > 1$ and $r \geq 1$, we take the same S defined in theorem 1.4 and let $n > rd^2$ with $d|n$. Then S is an index-r-free sequence for any $k \in$

$\left[0, \log_d^{\frac{n}{r}} -2\right)$ by theorem 1.4. Since $\lfloor \frac{m}{d^j} \rfloor > \frac{m}{d^j} - 1$, we calculate the length of S and have

$$\begin{aligned}|S| &= \sum_{(i,j)\in A} \left(\left\lfloor \frac{m}{d^j} \right\rfloor - (dr-1)d^{k-j} - 1\right) \\ &> \sum_{(i,j)\in[1,d-1]\times[0,k]} \left(\frac{m}{d^j} - (dr-1)d^{k-j} - 2\right) + m - (dr-1)d^k - 1 \\ &=(d-1)\sum_{j\in[0,k]} \left(\frac{m}{d^j} - (dr-1)d^{k-j} - 2\right) + m - (dr-1)d^k - 1 \\ &= \left(1 + \frac{1}{d} - \frac{1}{d^{k+1}}\right) n - (dr-1)(d^{k+1} + d^k - 1) - 2(k+1)(d-1) - 1\,.\end{aligned}$$

We let $k = \left\lfloor \frac{1}{2}\ln(n) \right\rfloor > 0$ and have

$$|S| > \left(1 + \frac{1}{d}\right) n + C_1\sqrt{n} + C_2 \ln(n) + C_3\,,$$

where C_1, C_2 and C_3 are some constants determined by d and r. Thus we have proved the theorem. □

Therefore, $\mathsf{t}_\mathbf{r}(n) \geq n + \frac{n}{d} + O(\sqrt{n})$ for every sufficiently large integer n divisible by d, where $d > 1$ and $r \geq 1$ are constant integers.

4. Concluding Remarks

Given any fixed integers $d > 1$ and $r \geq 1$. Since $\lfloor \frac{m}{d^j} \rfloor \leq \frac{m}{d^j}$, we can also get upper bounds of $|S|$ in theorem 1.5. Let d be the least prime factor of n. Generally, $|S| < n + \frac{n}{d}$. So we have the following conjecture.

Conjecture 4.1. *Let n be a composite number, C_n a cyclic group of order n, and d the least prime factor of n. Then every sequence S of length $|S| = n + \frac{n}{d}$ over C_n has a zero-sum subsequence T with* $\text{ind}(T) = 1$.

Open Problem. *Determine $\mathsf{t}_r(n)$ for all integers $n \geq 2$ and $r > 0$.*

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